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Maximal almost disjoint families and pseudocompactness of hyperspaces

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We show that all Ψ-spaces associated to maximal almost disjoint families have pseudocompact Vietoris hyperspace if and only if $\mathsf{MA}_{\mathfrak{c}}(\mathcal{P}(\omega)/\mathrm{fin})$ holds. We further study the question whether there is a maximal almost disjoint family whose hyperspace is pseudocompact and construct a consistent example of a maximal almost disjoint family of size $\omega_2 < \mathfrak{c}$ whose hyperspace is not pseudocompact.

1. Introduction and notation

Recall that an infinite collection $\mathcal{A} \subseteq [\omega]^\omega$ is *almost disjoint (AD)* if any two of its members have finite intersection. An AD family is *maximal (MAD)* if it is not properly contained in any other almost disjoint family.

Given an almost disjoint family $\mathcal{A}$, the *Mrówka–Isbell* space $\Psi(\mathcal{A})$ associated to $\mathcal{A}$ is the space $\omega \cup \mathcal{A}$, where ω is open and discrete and an open neighborhood basis for $A \in \mathcal{A}$ is $\{\{A\} \cup (A \setminus F) : F \in [\omega]^{<\omega}\}$. It is straightforward to verify that this is a Hausdorff, locally compact, first countable, non compact,

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zero dimensional topological space, and it is *pseudocompact* (every $\mathbb{R}$-valued continuous function on X is bounded), if and only if $\mathcal{A}$ is maximal (see e.g. [17]).

The *Vietoris hyperspace* of a topological space X is the set

$$\exp(X) = \{F \subseteq X : F \neq \emptyset \,\text{and}\, F \,\text{is closed}\}$$

endowed with the topology generated by the sets

$$U^- = \{F \in \exp(X) : F \cap U \neq \emptyset\} \text{ and}$$
$$U^+ = \{F \in \exp(X) : F \subseteq U\},$$

where $U \subseteq X$ is open.

In [13], J. Ginsburg proved that for a Tychonoff space X, if $\exp(X)$ is pseudocompact, then every finite power of X is also pseudocompact. He asked whether there is a relation between the pseudocompactness of X^ω and that of $\exp(X)$, and asked whether it is possible to characterize those spaces which have pseudocompact hyperspaces.

J. Cao, T. Nogura and A. Tomita [7] provided a partial answer by showing that for every homogeneous Tychonoff space X, if $\exp(X)$ is pseudocompact, then X^ω is pseudocompact. On the other hand, M. Hrušák, F. Hernández-Hernández and I. Martínez-Ruiz [17] showed that, in ZFC, there is a subspace of $\beta\omega$ containing ω such that X^ω is pseudocompact but $\exp(X)$ is not. This was extended by V. Rodrigues, A. Tomita and Y. Ortiz-Castillo [22], who showed that there is a space X such that X^κ is countably compact for every $\kappa < \mathfrak{h}$, but $\exp(X)$ is still not pseudocompact. They also showed that whenever X is a subspace of $\beta\omega$ containing ω, if $\exp(X)$ is pseudocompact, so are $\exp(X)^\omega$ and X^ω.

J. Cao and T. Nogura, in a private conversation, asked whether $\exp(X)$ is pseudocompact for some/every Mrówka–Isbell space X. The first relevant observation is:

Proposition 1.1 *([17]). Let $\mathcal{A}$ be an AD family. Then $\Psi(\mathcal{A})$ is pseudocompact iff $\Psi(\mathcal{A})^\omega$ is pseudocompact iff $\mathcal{A}$ is MAD.*

In particular, if $\mathcal{A}$ is an almost disjoint family and $X = \Psi(\mathcal{A})$, the following implications hold:

$$\exp(X) \text{ is pseudocompact} \implies X \text{ is pseudocompact} \implies X^\omega \text{ is pseudocompact}$$

So Ginsburg's questions restricted to the class of Mrówka-Isbell spaces becomes the problem of characterizing those MAD families such that the hyperspace of their Mrówka-Isbell space is pseudocompact. To study Ginsburg's question restricted to this class of spaces, the following shorthands will come in handy: if $\mathcal{A}$ is an almost disjoint family, then we define $\exp(A)$ as $\exp(\Psi(\mathcal{A}))$ and we call it the *hyperspace of* $\mathcal{A}$. We also say $\mathcal{A}$ is *pseudocompact* iff $\exp(\mathcal{A}) = \exp(\Psi(\mathcal{A}))$ is pseudocompact.

Recall that a family $\mathcal{P} \subseteq [\omega]^\omega$ is *centered* if the intersection of any finite number of members of $\mathcal{P}$ is infinite. A set $A \in [\omega]^\omega$ is a *pseudointersection* of $\mathcal{P}$ if $A \subseteq^* P$ (i.e. $A \setminus P$ is finite) for every $P \in \mathcal{P}$. The *pseudointersection number* $\mathfrak{p}$ is the smallest cardinality of a centered $\mathcal{C} \subseteq [\omega]^\omega$ with no pseudointersection. A collection $\mathcal{D} \subseteq [\omega]^\omega$ is *open dense* if for every $A \in [\omega]^\omega$ there exists $B \in \mathcal{D}$ such that $B \subseteq A$, and if for every $A \in [\omega]^\omega$ and for every $B \in \mathcal{D}$, if $A \subseteq^* B$ then $A \in \mathcal{D}$. The *distributivity number* $\mathfrak{h}$ is the least cardinality of a family of open dense subsets of $[\omega]^\omega$ with empty intersection.

The main result of [17] states:

Theorem 1.2 *([17]).*

(1) If $\mathfrak{p} = \mathfrak{c}$, then every MAD family is pseudocompact.

(2) If $\mathfrak{h} < \mathfrak{c}$, there is a MAD family which is not pseudocompact.

Part (2) of the theorem depends heavily on the *base tree theorem* of Balcar, Pelant and Simon [1] which affirms the existence of a base tree of height $\mathfrak{h}$, that is, of a tree $\mathcal{T} \subseteq [\omega]^\omega$ of height $\mathfrak{h}$ ordered by $\supseteq^*$, such that every element has $\mathfrak{c}$-many immediate successors, each level is a MAD family and such that every infinite subset of ω has a subset in the tree. As mentioned in [17], the assumption $\mathfrak{h} < \mathfrak{c}$ in (2) can be weakened to the existence of a base tree without branches of length $\mathfrak{c}$.

In [23], V. Rodrigues and A. Tomita showed that after adding ω_1 Cohen reals there is a Cohen indestructible MAD family of cardinality ω_1 whose hyperspace is pseudocompact.

In this article we optimize the above theorem by showing (Theorem 2.4) that the statement that all MAD families have pseudocompact hyperspace is equivalent to the assertion $\mathsf{MA}_{\mathfrak{c}}(\mathcal{P}(\omega)/\text{fin})$.[5]

The problem of whether there is a pseudocompact MAD family in ZFC was raised in [17] and is still open:

Question 1.3. *Is there a MAD family $\mathcal{A}$ with pseudocompact hyperspace in ZFC?*

Here we provide a partial answer to the problem by showing that it is consistent that there is a MAD family $\mathcal{A}$ of size strictly less than $\mathfrak{c}$ whose hyperspace is not pseudocompact, so, in particular, there is an AD family of size less than $\mathfrak{c}$ which cannot be extended to a pseudocompact one, i.e. it is consistent that pseudocompact MAD families do not exist *generically*.

Our notation is mostly standard. In particular, ω denotes the set of finite von Neumann ordinals and is identified with the natural numbers. The set of free ultrafilters over ω is denoted by ω^* and is identified with the remainder of the Stone-Čech compactification of ω. Given $\mathcal{U} \in \omega^*$, a topological space X, $x \in$ and a sequence $\langle x_n : n \in \omega \rangle$ of elements of X, we say that x is a *$\mathcal{U}$-limit* of $\langle x_n : n \in \omega \rangle$ if for every neighborhood U of x, the set $\{n \in \omega : x_n \in U\}$ belongs to $\mathcal{U}$ and we then write $\mathcal{U}\text{-}\lim x_n = x$.

The smallest cardinality of a MAD family is defined as $\mathfrak{a}$. It is well known that $\omega_1 \leq \mathfrak{p} \leq \mathfrak{h} \leq \mathfrak{a} \leq \mathfrak{c}$ and that all inequalities are consistently strict. See [2] for more on cardinal invariants of the continuum.

2. Equivalence with $\mathbf{MA}_{\mathfrak{c}}(\mathcal{P}(\omega)/\,\text{fin})$

In this section we shall identify statements equivalent to the assertion "For every MAD family $\exp(\mathcal{A})$ is pseudocompact".

The following proposition appears as Proposition 2.1 in [23]:

Proposition 2.1. *Let $\mathcal{A}$ be an almost disjoint family. Then $\exp(\mathcal{A})$ is pseudocompact if and only if every sequence $\langle a_n : n \in \omega \rangle \subseteq [\omega]^{<\omega} \setminus \{\emptyset\} \subseteq \exp(\mathcal{A})$ of pairwise disjoint sets has an accumulation point in $\exp(\mathcal{A})$.*

By using this proposition we can get a result similar to Lemma 3.1 in [17].

Lemma 2.2. *Let $\mathcal{A}$ be an almost disjoint family. Let $F = \langle F_n : n \in \omega \rangle$ be a sequence of pairwise disjoint finite nonempty subsets of ω. Given $A \subseteq \omega$, let $I_A = \{n \in \omega : F_n \cap A \neq \emptyset\}$ and $M_A = \{n \in \omega : F_n \subseteq A\}$. Then:*

(1) If L is a limit point of the sequence F in $\exp(\mathcal{A})$, then $L \subseteq \mathcal{A}$, and

[5] If κ is a cardinal and P is a pre-order, $\mathrm{MA}_\kappa(P)$ is the statement "for every collection of $\leq \kappa$ dense subsets of P there exists a filter G on P which intersects every dense set of the collection". The boolean algebra $\mathcal{P}(\omega)/\,\text{fin}$ can be seen as the set $[\omega]^\omega$ ordered by $\subseteq^*$.

(2) Given $L \subseteq \mathcal{A}$, L is a limit point of F if, and only if for every $P \subseteq \omega$ such that $\forall A \in L\, A \subseteq P$, the set $\{I_A : A \in L\} \cup \{M_P\}$ is centered.

Proof. For the first item, notice that if $n \in \omega \cap L$, then $\{n\}^-$ is a neighborhood of L which intersects at most one element from the sequence F, so L cannot be a limit point for F.

For the second item, first suppose that L is a limit point of F. Fix arbitrary $A_0, \dots, A_l \in L$ and P as in the item. We must show that $I_{A_0} \cap \dots \cap I_{A_n} \cap M_P$ is infinite. Fix $k \in \omega$. Notice that $L \cup (P \setminus k)$ is open, so $V = (L \cup P)^+ \cap (\{A_0\} \cup A_0)^- \cdots \cap (\{A_n\} \cup A_l)^-$ is a neighborhood of L, so it must have a point F_n with $n \geq k$. Then $F_n \subseteq P$ and $F_n \cap A_i \neq \emptyset$ for each i, that is, $n \in I_{A_0} \cap \dots \cap I_{A_l} \cap M_P \setminus k$. Since k is arbitrary we are done.

Now we prove the converse. Let $U_0, \dots, U_n, V$ be open sets of $\Psi(\mathcal{A})$ such that $L \in U_0^- \cap \dots \cap U_n^- \cap V^+$. Let $P = V \cap \omega$ and, for each $i \leq l$, let $A_i \in L \cap U_i$ and let k_i be such that $A_i \setminus k_i \subseteq U_i$. Then $I_{A_0} \cap \dots \cap I_{A_l} \cap M_P$ is infinite. Since F is a pairwise disjoint sequence, there exists m such that for all $n \geq m$, $F_n \cap \max\{k_0, \dots, k_l\} = \emptyset$. Let $m \geq n$ be in $I_{A_0} \cap \dots \cap I_{A_l} \cap M_P$. Then $F_m \in U_0^- \cap \dots \cap U_l^- \cap V^+$ and the proof is complete. □

A sufficient condition to guarantee the existence of a limit point is given by the following lemma:

Lemma 2.3. *Let $\mathcal{A}$ be an almost disjoint family, $\mathcal{U}$ be a free ultrafilter and let $F = \langle F_n : n \in \omega \rangle \subseteq [\omega]^{<\omega} \setminus \{\emptyset\} \subseteq \exp(\mathcal{A})$ be a sequence of pairwise disjoint sets. Then if for every $f \in \prod_{n\in\omega} F_n$ there exists $A \in \mathcal{A}$ and $B \in \mathcal{U}$ such that $f[B] \subseteq A$, then F has a $\mathcal{U}$-limit.*

Proof. Let $P = \prod_{n\in\omega} F_n$. Given $f \in P$, fix $B_f \in \mathcal{U}$ and $A_f \in \mathcal{A}$ such that $f[B_f] \subseteq A_f$. Let $\mathcal{B} = \{A_f : f \in P\}$ We claim that $\mathcal{B} = \mathcal{U}\text{-}\lim F$.

To verify the claim, it suffices to verify the $\mathcal{U}$-limit condition for sub-basic sets, so let $U \subset \Psi(\mathcal{A})$ be open.

If $\mathcal{B} \in U^-$, then there exists $f \in P$ with $A_f \in U$. Since U is open, $A_f \subseteq^* U$. Then $f[B_f] \subseteq^* U$. So $B_f \subseteq^* \{n \in \omega : f(n) \in U\} \subseteq \{n \in \omega : F_n \in U^-\}$. Since $B_f \in \mathcal{U}$ and $\mathcal{U}$ is a free ultrafilter, it follows that $\{n \in \omega : F_n \in U^-\} \in \mathcal{U}$.

If $\mathcal{B} \in U^+$, suppose by contradiction that $\{n \in \omega : F_n \in U^+\} \notin \mathcal{U}$. Then $I = \{n \in \omega : F_n \setminus U \neq \emptyset\} \in \mathcal{U}$. Let $f \in P$ be such that for each $n \in I$, $f(n) \in F_n \setminus U$. Then $f[I \cap B_f] \subseteq^* A_f$ and $f[I \cap B_f] \setminus U$ is infinite, so $A_f \setminus U$ is infinite. On the other hand, since $\mathcal{B} \in U^+$ we have $A_f \in U$, but U is open, so $A_f \subseteq^* U$, a contradiction. □

Given a T_1 topological space X with no isolated points, the *Baire number* of X, denoted by $\mathfrak{n}(X)$, is the smallest cardinality of a family of open dense subsets of X with empty intersection. In the following theorem, the equivalence between a) and d) with an arbitrary infinite κ in the place of $\mathfrak{c}$ was presented without proof in [1]. For the sake of completeness, we present a proof (in the proof we present, one could switch $\mathfrak{c}$ for any other infinite cardinal).

Theorem 2.4. *The following are equivalent:*

a) $\mathrm{MA}_{\mathfrak{c}}(\mathcal{P}(\omega)/\mathrm{fin})$
b) For every MAD family $\mathcal{A}$, $\exp(\mathcal{A})$ is pseudocompact,
c) $\mathfrak{h} = \mathfrak{c}$ and every base tree has a cofinal branch
d) $\mathfrak{n}(\omega^*) > \mathfrak{c}$.

Proof. $a) \to b)$ Suppose $\mathrm{MA}_{\mathfrak{c}}(\mathcal{P}(\omega)/\mathrm{fin})$ holds and fix a MAD family $\mathcal{A}$. Let $F = \langle F_n : n \in \omega \rangle \subseteq [\omega]^{<\omega} \setminus \{\emptyset\}$ be a sequence of pairwise disjoint sets. Let $P = \prod_{n\in\omega} F_n$. Given $f \in P$, let

$$D_f = \{B \in [\omega]^\omega : \exists A \in \mathcal{A}\, f[B] \subseteq A\}.$$

It is straightforward to verify that D_f is dense in $\mathcal{P}(\omega)/\text{fin}$. By $\text{MA}_{\mathfrak{c}}(\mathcal{P}(\omega)/\text{fin})$, let $\mathcal{U}$ be a filter intersecting every member of $\{D_f : f \in P\}$. Then, by Lemma 2.3, F has a $\mathcal{U}$-limit. Now the conclusion follows from Proposition 2.1.

b) → *c*) Negating *c*), either $\mathfrak{h} < \mathfrak{c}$ or there exists a base tree of height $\mathfrak{c}$ with no cofinal branches. Either way, there is a base tree with no branches of cardinality $\mathfrak{c}$, so the negation of *b*) follows from the second statement of Theorem 1.2 and from the comments below it.

c) → *d*) Let $(U_\alpha : \alpha < \mathfrak{c})$ be a collection of open dense subsets of ω^* (where ω^* is identified with the space of free ultrafilters on ω. For each α, let $\mathcal{A}_\alpha$ be an infinite almost disjoint family such that $A^* \subseteq U_\alpha$ for every $A \in \mathcal{A}_\alpha$ maximal for this property. It is easy to verify that each $\mathcal{A}_\alpha$ is a MAD family. Using $\mathfrak{h} = \mathfrak{c}$ and following the standard construction of a base tree (e.g. [2]), there exists a base tree $\mathcal{T}$ of height $\mathfrak{c}$ such that every level $\mathcal{T}_\alpha$ of $\mathcal{T}$ refines every element of $\{\mathcal{A}_\beta : \beta < \alpha\}$ (that is: given $\beta < \alpha$ and $A \in \mathcal{T}_\alpha$, there exists $B \in \mathcal{A}_\beta$ such that $A \subseteq^* B$). Then $\mathcal{T}$ has a cofinal branch $\mathcal{T}$. Extend $\mathcal{T}$ to an ultrafilter $\mathcal{U}$. $\mathcal{U}$ intersects $\mathcal{T}_\alpha$ for every $\alpha < \mathfrak{c}$, so it would also intersect $\mathcal{A}_\alpha$ for every $\alpha < \mathfrak{c}$. This shows that $\mathcal{U} \in \bigcap_{\alpha<\mathfrak{c}} U_\alpha$.

d) → *a*) Suppose $\mathfrak{n}(\omega^*) > \mathfrak{c}$ and let $(\mathcal{B}_\alpha : \alpha < \mathfrak{c})$ be a collection of dense subsets of $\mathcal{P}(\omega)/\text{fin}$. For each $\alpha < \mathfrak{c}$, let $U_\alpha = \bigcup\{B^* : B \in \mathcal{B}_\alpha\}$. It is easy to verify U_α is open and dense in ω^*. Let $\mathcal{U} \in \bigcap_{\alpha<\mathfrak{c}} U_\alpha$. Then for each $\alpha < \mathfrak{c}$ there exists $B \in \mathcal{B}_\alpha$ such that $\mathcal{U} \in B^*$, that is, $B \in \mathcal{U} \cap \mathcal{B}_\alpha$, i.e. $\mathcal{U}$ is generic for $(\mathcal{B}_\alpha : \alpha < \mathfrak{c})$. □

Next we present a model of $\mathfrak{p} < \mathfrak{c}$ where all Mrówka-Isbell spaces from MAD families have pseudocompact hyperspaces.

Theorem 2.5. *It is consistent that* $\mathfrak{p} < \mathfrak{c}$ *and* $\exp(\mathcal{A})$ *is pseudocompact for every MAD family* $\mathcal{A}$.

Proof. Suppose $V \vDash \mathfrak{p} = \mathfrak{c} = \omega_2 +$ there exists a Suslin Tree. Let S be a well-pruned Suslin tree and let G be S generic over $\mathbf{V}$. It is well known that S forces $\mathfrak{p} = \omega_1 < \mathfrak{c}$ (see, for example, [10]). Suppose $\mathcal{A}$ is a MAD family in $V[G]$.

Claim. *There exists a MAD family* $\mathcal{B} \in \mathbf{V}$ *such that for every* $B \in \mathcal{B}$ *there exists* $A \in \mathcal{A}$ *such that* $B \subseteq^* A$.

Proof of the claim. Let $\mathring{A}$ be a name for $\mathcal{A}$ and let $p \in S$ be such that $p \Vdash \mathring{A}$ is a MAD family. If $t \leq p$, let $\mathcal{A}_t = \{A \in [\omega]^\omega : t \Vdash \check{A} \in \mathring{A}\}$. Each of these sets is an almost disjoint family. In $\mathbf{V}$, for each $t \leq p$ let $\mathcal{B}_t$ be a MAD family containing $\mathcal{A}_t$.

Since $|S| = \omega_1 < \mathfrak{h}$, there exists $\mathcal{B}$ refining $\{\mathcal{B}_t : t \leq p\}$, that is, for every $B \in \mathcal{B}$ and for every $t \leq p$, there exists $A \in \mathcal{B}_t$ such that $B \subseteq^* A$.

We show that $\mathcal{B}$ is as intended: given $B \in \mathcal{B}$, there exists $A \in \mathcal{A}$ such that $|B \cap A| = \omega$. Since forcing with a Suslin trees does not add reals, there exists $t \leq p$ such that $t \Vdash A \in \mathring{A}$, so $A \in \mathcal{A}_t$. There exists $A' \in \mathcal{B}_t$ such that $B \subseteq^* A'$. Since $A', A \in \mathcal{B}_t$, it follows that $A = A'$, which completes the proof of the claim. □

Let $F \in \mathbf{V}[G]$ be a sequence of pairwise disjoint finite nonempty subsets of ω. Since forcing with S does not add reals, $F \in \mathbf{V}$. Working in $\mathbf{V}$, since $\mathfrak{p} = \mathfrak{c}$ holds, there exists a free ultrafilter $\mathcal{U}$ such that for every $f \in \prod_{n\in\omega} F_n$ there is $I \in \mathcal{U}$ such that $f[I]$ is contained in an element of $\mathcal{B}$.

In $\mathbf{V}[G]$, $\mathcal{U}$ is still a free ultrafilter and for every $f \in \prod_{n\in\omega} F_n$ there is $I \in \mathcal{U}$ such that $f[I]$ is contained in an element of $\mathcal{A}$. This implies that every such an f has a $\mathcal{U}$-limit in $\Psi(\mathcal{A})$ and that in the hyperspace, $\mathcal{U}\text{-lim}\, F = \{\mathcal{U}\text{-}\lim f : f \in \prod_{n\in\omega} F_n\}$. □

3. Generic existence of pseudocompact MAD families

In this section we study sufficient conditions for the existence of pseudocompact MAD families. We give sufficient conditions for the existence of both large and small pseudocompact MAD families. Following [14] we shall say that *pseudocompact MAD families exist generically* if every AD family of size less than $\mathfrak{c}$ can be extended to a pseudocompact one. Of course, it trivially follows from the results of the previous section that pseudocompact MAD families exist generically if the conditions of Theorem 2.4 are satisfied, i.e. if $\mathfrak{h} = \mathfrak{c}$ and every base tree has a cofinal branch.

On the other hand, this is not equivalent to the generic existence of pseudocompact MAD families which we shall show next. Recall [6] that given an ultrafilter $\mathcal{U}$ the *pseudointersection number* $\mathfrak{p}(\mathcal{U})$ of $\mathcal{U}$ is defined as the minimal size of a subfamily $\mathcal{X}$ of $\mathcal{U}$ without a pseudointersection in $\mathcal{U}$, i.e. $\mathfrak{p}(\mathcal{U}) > \omega$ if and only if $\mathcal{U}$ is a *P-point*, and $\mathfrak{p}(\mathcal{U}) = \mathfrak{c}$ if and only if $\mathcal{U}$ is a *simple $P_\mathfrak{c}$-point* i.e. an ultrafilter generated by a $\subseteq^*$-decreasing chain of length $\mathfrak{c}$.

Theorem 3.1. *If $\mathcal{A}$ is a MAD family, $\mathcal{U}$ an ultrafilter and $|\mathcal{A}| < \mathfrak{p}(\mathcal{U})$ then $\mathcal{A}$ is pseudocompact.*

Proof. Let $\mathcal{U}$ be given. Fix a MAD family $\mathcal{A}$ such that $|\mathcal{A}| < \mathfrak{p}(\mathcal{U})$. By Lemma 2.3, and Proposition 2.1, it is sufficient to verify that for every injective sequence $f : \omega \to \omega$ there exists $B \in \mathcal{U}$ and $A \in \mathcal{A}$ such that $f[B] \subseteq A$.

Suppose this is not the case. Then there exists $f : \omega \to \omega$ such that for all $A \in \mathcal{A}$ and $B \in \mathcal{U}$, $f[B] \setminus A$ is infinite. First, notice that given $A \in \mathcal{A}$, there exists $B_A \in \mathcal{U}$ such that $f[B_A] \cap A$ is empty: the sets $\{n \in \omega : f(n) \notin A\}$ and $\{n \in \omega : f(n) \in A\}$ form a partition of ω, so one of them is in $\mathcal{U}$. But the second is not in $\mathcal{U}$ by hypothesis. Let B_A be the first set.

Now let B be a pseudointersection of $\{B_A : A \in \mathcal{A}\}$ in $\mathcal{U}$. It follows that $f[B] \cap A$ is finite for every $A \in \mathcal{A}$, contradicting the maximality of $\mathcal{A}$. □

Note that the same argument shows that:

Corollary 3.2. *If there is an ultrafilter $\mathcal{U}$ such that $\mathfrak{p}(\mathcal{U}) = \mathfrak{c}$ then pseudocompact MAD families exist generically.*

Next we will construct a model where the assumptions of Theorem 3.1 hold, i.e. $\mathfrak{a} = \omega_1$ and there is an ultrafilter $\mathcal{U}$ such that $\mathfrak{p}(\mathcal{U}) = \omega_2$. We will use the method of matrix iterations, which was introduced by Blass and Shelah in [3] and further developed by Brendle and Fischer in [5]. We will provide a quick review of this method, but it would be helpful if the reader had familiarity with [5]. To learn more about matrix iterations, the reader may consult [20,12,11,4,9,8].

The following forcing was introduced by Hechler [15] for adding generically a MAD family (see also [5]). Let $\gamma \leq \omega_1$. Define $\mathbb{H}_\gamma$ as the set of all functions p such that there are $F_p \in [\gamma]^{<\omega}$ and $n_p \in \omega$ such that $p : F_p \times n_p \longrightarrow 2$.

Given $p, q \in \mathbb{H}_\gamma$, define $p \leq q$ if the following holds:

(1) $q \subseteq p$ (hence $F_q \subseteq F_p$ and $n_q \leq n_p$).
(2) For every $\alpha, \beta \in F_q$ (with $\alpha \neq \beta$) and $i \in [n_q, n_p)$, if $p(\alpha, i) = 1$, then $p(\beta, i) = 0$.

Assume $G \subseteq \mathbb{H}_\gamma$ is a generic filter. For every $\alpha < \gamma$, define

$$A^G_\alpha = \{i \mid p \in G\,(p(\alpha, i) = 1)\}.$$

Define the *generic AD family* as $\mathcal{A}^G_\gamma = \left\{A^G_\alpha \mid \alpha < \gamma\right\}$. The following lemma is well known and easy to see:

Lemma 3.3. *Let $\gamma \leq \omega_1$ and $G \subseteq \mathbb{H}_\gamma$ a generic filter.*

(1) If $\alpha < \gamma$, then A^G_α is infinite.
(2) $\mathcal{A}^G_\gamma$ is an AD family.

(3) If $\delta < \gamma$, then $\mathbb{H}_\delta$ is a regular suborder of $\mathbb{H}_\gamma$.
(4) If $\gamma = \omega_1$, then $\mathcal{A}^G_{\omega_1}$ is a MAD family.

More properties and preservation results may be consulted in [19].

Let $\mathcal{F}$ be a filter on ω. Define the *Mathias forcing of* $\mathcal{F}$ (denoted as $\mathbb{M}(\mathcal{F})$) [18] as the set of all $p = (s_p, F_p)$ such that $s_p \in [\omega]^{<\omega}$ and $F_p \in \mathcal{F}$, ordered by $p = (s_p, F_p) \leq q = (s_q, F_q)$ if $s_q \subseteq s_p$, $F_p \subseteq F_q$ and $s_p \setminus s_q \subseteq F_q$.

If $G \subseteq \mathbb{M}(\mathcal{F})$ is a generic filter, the *generic real of* $\mathbb{M}(\mathcal{F})$ is defined as

$$r_G = \bigcup \{s_p \mid \exists p = (s_p, F_p) \in G\}.$$

It is easy to see that r_G is a pseudointersection of $\mathcal{F}$.

The following notion was introduced in [5]:

Let $M \subseteq N$ be transitive models of ZFC (we may assume that N is a forcing extension of M). Let $\mathcal{A} = \{A_\alpha \mid \alpha \in \gamma\}$ be an AD family in M and $B \in N$ an infinite subset of ω. We say that $\bigstar^{M,N}_{\mathcal{A},B}$ holds, if for all $h : \omega \times [\gamma]^{<\omega} \to \omega$ such that $h \in M$, for all $m \in \omega$ and for all $F \in [\gamma]^{<\omega}$, there exists $n \geq m$ such that $[n, h(n, F)) \setminus \bigcup_{\alpha \in F} A_\alpha \subseteq B$. It is easy to see that if $\bigstar^{M,N}_{\mathcal{A},B}$ hold, then $B \in \mathcal{I}(\mathcal{A})^+$.

The following is immediate from the definition:

Lemma 3.4. *Let $M \subseteq N$ be transitive models of* ZFC, *$\mathcal{A} = \{A_\alpha \mid \alpha \in \gamma\} \in M$ an AD family and $B \in N$ such that $\bigstar^{M,N}_{\mathcal{A},B}$ holds. If $X \in \mathcal{I}(\mathcal{A})^+ \cap M$ then $B \cap X$ is infinite (in N).*

The next lemma is Lemma 4 of [5]:

Lemma 3.5 *([5]). Let $\gamma + 1 \leq \omega_1$ and $G_{\gamma+1} \subseteq \mathbb{H}_{\gamma+1}$ a generic filter. Define $G_\gamma = \mathbb{H}_\gamma \cap G_{\gamma+1}$. Then $\bigstar^{V[G_\gamma],V[G_{\gamma+1}]}_{\mathcal{A}_\gamma,A_\gamma}$ holds.*

The following is a deep result of Brendle and Fischer (Crucial Lemma 7 of [5]):

Proposition 3.6 *(Brendle, Fischer [5]). Let $M \subseteq N$ be transitive models of* ZFC, *$\mathcal{A} = \{A_\alpha \mid \alpha \in \gamma\} \in M$ an AD family and $B \in N$ such that $\bigstar^{M,N}_{\mathcal{A},B}$ holds. Let $\mathcal{U} \in M$ be an ultrafilter. There is an ultrafilter $\mathcal{W} \in N$ such that the following holds:*

(1) $\mathcal{U} \subseteq \mathcal{W}$ (hence $\mathbb{M}(\mathcal{U}) \subseteq \mathbb{M}(\mathcal{W})$).
(2) If $L \subseteq \mathbb{M}(\mathcal{U})$ is a maximal antichain with $L \in M$, then L is also a maximal antichain of $\mathbb{M}(\mathcal{W})$.
(3) If $G_\mathcal{W} \subseteq \mathbb{M}(\mathcal{W})$ is an $(N, \mathbb{M}(\mathcal{W}))$-generic filter, then $G_\mathcal{U} = G_\mathcal{W} \cap \mathbb{M}(\mathcal{U})$ is an $(M, \mathbb{M}(\mathcal{U}))$-generic filter.
(4) $r_{G_\mathcal{W}} = r_{G_\mathcal{U}}$ (in particular, $r_{G_\mathcal{W}} \in M[G_\mathcal{U}]$, but this does not imply that an $\mathbb{M}(\mathcal{U})$-generic real is also a $\mathbb{M}(\mathcal{W})$-generic real).
(5) $\bigstar^{M[G_\mathcal{U}],\, N[G_\mathcal{W}]}_{\mathcal{A},B}$ holds.

Note that points 3 and 4 follow from points 1 and 2. It is important to note that in general (in N) $\mathbb{M}(\mathcal{U})$ will not be a regular suborder of $\mathbb{M}(\mathcal{W})$ (except in the trivial case where $\mathcal{U} = \mathcal{W}$). This is because in point 2, we only have the results for the maximal antichains that are in M, but it may fail for those that are in N.

Let κ and λ be two cardinals. We will say that

$$\left(\langle \mathbb{P}_{\alpha,\beta} \mid \alpha \leq \kappa, \beta \leq \lambda\rangle, \langle \dot{\mathbb{Q}}_{\alpha,\beta} \mid \alpha \leq \kappa, \beta < \lambda\rangle\right)$$

is a *standard matrix iteration* if the following holds for every $\alpha \leq \kappa, \beta \leq \lambda$:

(1) If $\beta < \lambda$, then $\dot{\mathbb{Q}}_{\alpha,\beta}$ is a $\mathbb{P}_{\alpha,\beta}$-name for a partial order with the countable chain condition.
(2) If $\beta < \lambda$, then $\mathbb{P}_{\alpha,\beta+1} = \mathbb{P}_{\alpha,\beta} * \dot{\mathbb{Q}}_{\alpha,\beta}$.
(3) If $\xi < \beta$, then $\mathbb{P}_{\alpha,\xi}$ is a regular suborder of $\mathbb{P}_{\alpha,\beta}$.
(4) If β is limit, then $\mathbb{P}_{\alpha,\beta}$ is the finite support iteration of $\langle \mathbb{P}_{\alpha,\xi} \mid \xi < \beta \rangle$.
(5) If $\eta < \alpha$, then $\mathbb{P}_{\eta,\beta}$ is a regular suborder of $\mathbb{P}_{\alpha,\beta}$.
(6) If α is limit, then $\mathbb{P}_{\alpha,0}$ is the finite support iteration of $\langle \mathbb{P}_{\eta,0} \mid \eta < \alpha \rangle$.
(7) If $p \in \mathbb{P}_{\kappa,\beta}$, then there is $\gamma < \kappa$ such that $p \in \mathbb{P}_{\gamma,\beta}$.
(8) If $\dot{f}$ is a $\mathbb{P}_{\kappa,\beta}$-name for a real, then there is $\gamma < \kappa$ such that $\dot{f}$ is a $\mathbb{P}_{\gamma,\beta}$-name.

In the above situation, given $\alpha \leq \kappa, \beta \leq \lambda$, we denote by $V_{\alpha\beta}$ the extension of V by forcing with $\mathbb{P}_{\alpha,\beta}$.

We now define $\left(\langle \mathbb{P}_{\alpha,\beta} \mid \alpha \leq \omega_1, \beta \leq \omega_2 \rangle, \langle \dot{\mathbb{Q}}_{\alpha,\beta} \mid \alpha \leq \omega_1, \beta < \omega_2 \rangle\right)$ such that for every $\alpha \leq \omega_1$ and $\beta \leq \omega_2$ we have the following properties:

(1) $\mathbb{P}_{\alpha 0} = \mathbb{H}_\alpha$.
(2) Let $\mathcal{A}_\alpha = \{A_\xi \mid \xi < \alpha\}$ be the AD family added by $\mathbb{H}_\alpha$.
(3) For every $\beta < \omega_2$, there is a sequence $\langle \mathcal{U}_{\gamma\beta} \mid \gamma \leq \omega_1 \rangle$ with the following properties:
    (a) $\mathcal{U}_{\gamma\beta} \in V_{\gamma\beta}$ and it is an ultrafilter in such model.
    (b) For every $\gamma < \delta \leq \omega_1$ the following holds:
        (i) $\mathcal{U}_{\gamma\beta} \subseteq \mathcal{U}_{\delta\beta}$.
        (ii) If $L \subseteq \mathbb{M}(\mathcal{U}_{\gamma\beta})$ is a maximal antichain with $L \in V_{\gamma\beta}$, then L is also a maximal antichain of $\mathbb{M}(\mathcal{U}_{\delta\beta})$.
        (iii) If $\bigstar^{V_{\gamma\beta}, V_{(\gamma+1)\beta}}_{\mathcal{A}_\gamma, A_\gamma}$ and H is a $\mathbb{M}\left(\mathcal{U}_{(\gamma+1)\beta}\right)$-generic filter over $V_{(\gamma+1)\beta}$, then $\bigstar^{V_{\gamma\beta}[H], V_{(\gamma+1)\beta}[H]}_{\mathcal{A}_\gamma, A_\gamma}$.
(4) If $\beta < \omega_2$, then $\mathbb{P}_{\alpha,\beta} \Vdash$ "$\dot{\mathbb{Q}}_{\alpha,\beta} = \dot{\mathbb{M}}\left(\mathcal{U}_{\alpha\beta}\right)$" and $\mathbb{P}_{\alpha,\beta+1} = \mathbb{P}_{\alpha,\beta} * \dot{\mathbb{M}}\left(\mathcal{U}_{\alpha\beta}\right)$.
(5) If $\beta < \omega_2$ and r_β is the $\mathbb{M}\left(\mathcal{U}_{\omega_1\beta}\right)$-generic real over $V_{\omega_1\beta}$, then $r_\beta \in \mathcal{U}_{0(\beta+1)}$.
(6) If $\beta < \omega_2$ is a limit ordinal, then $\{r_\eta \mid \eta < \beta\} \subseteq \mathcal{U}_{0\beta}$.

By the construction, it follows that $\{r_\beta \mid \beta < \omega_2\}$ is a $\subseteq^*$-decreasing sequence (this is why point 6 makes sense). The main point is, of course, that the just defined

$$\left(\langle \mathbb{P}_{\alpha,\beta} \mid \alpha \leq \omega_1, \beta \leq \omega_2 \rangle, \langle \dot{\mathbb{Q}}_{\alpha,\beta} \mid \alpha \leq \omega_1, \beta < \omega_2 \rangle\right)$$

is a standard matrix iteration. This follows by the same arguments as in [3] or [5].

There is a subtle point that we would like to clarify in (5) and (6) above. Let $\beta < \omega_2$, in point (5) we demand that the $(V_{\omega_1\beta}, \mathbb{M}(\mathcal{U}_{\omega_1\beta}))$-generic real r_β is in $\mathcal{U}_{0(\beta+1)}$. In particular, we need that r_β belongs to $V_{0(\beta+1)}$. At first glance, this might seem impossible since (in principle) r_β is not an $\mathbb{M}(\mathcal{U}_{0\beta})$-name. However, this is easily fixed as follows: we simply require that $\mathcal{U}_{0(\beta+1)}$ contains the generic real added by $\mathbb{M}(\mathcal{U}_{0\beta})$, which we will denote by $r_{0\beta}$. By point 4 of Proposition 3.6, we get that r_β and $r_{0\beta}$ are equal (and in particular, r_β belongs to $\mathcal{U}_{0(\beta+1)}$). A similar remark applies to point (6). The same argument was used in [3]. We leave the rest of the details to the reader.

We can now prove the following:

Theorem 3.7. *There is a model of* ZFC *in which* $\mathfrak{a} = \omega_1$ *and there is an ultrafilter* $\mathcal{W}$ *such that* $\mathfrak{p}(\mathcal{W}) = \mathfrak{c} = \omega_2$.

Proof. We start with a model V of the Continuum Hypothesis. Let $G \subseteq \mathbb{P}_{\omega_1,\omega_2}$ be a generic filter (where $\mathbb{P}_{\omega_1,\omega_2}$ is the forcing described above). We will show that $V[G]$ is the model we are looking for. A straightforward argument shows that $V[G] \models \mathfrak{c} = \omega_2$.

We argue in $V[G]$. Note that $R = \{r_\beta \mid \beta < \omega_2\}$ is a decreasing tower, so it is centered. Let $\mathcal{W}$ be the filter generated by R. It is easy to see that $\mathcal{W}$ is in fact an ultrafilter (this is because r_β is a $\mathbb{M}(\mathcal{U}_{\omega_1\beta})$-generic real, for more details, the reader may consult [3]). Furthermore, since $\mathcal{W}$ is generated by a tower of length ω_2, it follows that $\mathfrak{p}(\mathcal{U}) = \omega_2$.

It remains to be proved that $\mathfrak{a} = \omega_1$ holds in $V[G]$. This is the same argument as the one used in section 4 of [5]. We include the argument for completeness. We will prove that $\mathcal{A}_{\omega_1} = \{A_\alpha \mid \alpha < \omega_1\}$ is a MAD family in $V[G]$. We start with the following:

Claim. *Let $\alpha < \omega_1$ and $\beta \leq \omega_2$. Then $\bigstar^{V_{\alpha\beta},V_{(\alpha+1)\beta}}_{\mathcal{A}_\alpha,\, A_\alpha}$ holds.*

Fix $\alpha < \omega_1$, we prove the claim by induction on β. The case $\beta = 0$ follows by Lemma 3.5. If the claim is true for $\beta < \omega_2$, then it is also true for $\beta + 1$ by point 3(b)iii in the definition of our iteration. Finally, let β be a limit ordinal and assume that the lemma is true for every ordinal less than β. If β has uncountable cofinality, then there is nothing to prove. For every $F \in [\alpha]^{<\omega}$, there is $\eta < \beta$ such that $h_F \in V_{\alpha\eta}$). If β has countable cofinality, the claim follows by the Lemma 12 point 1 of [5]). This proves the claim.

Claim. *$\mathcal{A}_{\omega_1}$ is a MAD family in $V[G]$.*

Let $X \in \mathcal{I}(\mathcal{A}_{\omega_1})^+$ (in $V[G]$). Since $\mathbb{P}_{\omega_1,\omega_2}$ is a finite support iteration of the c.c.c. partial orders $\langle \mathbb{P}_{\omega_1,\beta} \mid \beta < \omega_2\rangle$, there is $\beta < \omega_2$ such that $X \in V_{\omega_1\beta}$. Furthermore, since we are using a standard matrix iteration and X is a real, there is $\alpha < \omega_1$ such that $X \in V_{\alpha\beta}$. Since $\bigstar^{V_{\alpha\beta},V_{(\alpha+1)\beta}}_{\mathcal{A}_\alpha,\, A_\alpha}$ holds and $X \in \mathcal{I}(\mathcal{A}_\alpha)^+$, by Lemma 3.4, we have that $A_\alpha \cap X$ is infinite. This finishes the proof. □

4. Non-pseudocompact MAD families

Here we prove that consistently there is a MAD family $\mathcal{A}$ of size $< \mathfrak{c}$ which is not pseudocompact. This, of course, trivially provides a model where pseudocompact MAD families do not exist generically. The generic existence expresses the fact that a "naïve" construction of an object with the desired properties can be carried out, meaning that we line up all possible requirements (necessarily of length $\mathfrak{c}$) and try to fulfill them one by one without doing anything else to keep the recursion artificially alive. In this sense, Theorem 4.2 points out that if there is a pseudocompact MAD family in ZFC, its construction cannot be too simple and some further sophistication is required.

The example we construct will be a MAD family over the countably infinite set $\triangle = \{(n,m) \in \omega \times \omega : m \leq n\}$. The elements of $\mathcal{A}$ will be graphs of partial functions. The result easily follows from the following:

Theorem 4.1. *It is consistent with $\mathfrak{c} > \omega_2$ that there is a MAD family $\mathcal{A}$ of size ω_2 on $\triangle$ consisting of partial functions below the diagonal, and there are MAD families $\{\mathcal{A}_\alpha : \alpha < \omega_1\}$ on ω, such that*

(1) $\forall s \in \mathcal{A}\ \exists \alpha < \omega_1\ \mathrm{dom}(s) \in \mathcal{A}_\alpha$,
(2) $s \neq t \in \mathcal{A} \Rightarrow \mathrm{dom}(s) \neq \mathrm{dom}(t)$, *and*
(3) for every family $\mathcal{F}$ of ω_1-many partial functions below the diagonal there is a total function below the diagonal almost disjoint from all elements of $\mathcal{F}$.

We shall postpone the proof of the theorem and first show that it suffices to prove the desired result.

Theorem 4.2. *It is relatively consistent with ZFC that there is a non-pseudocompact MAD family $\mathcal{A}$ of size $< \mathfrak{c}$.*

Proof. Assume that $\mathfrak{c} > \omega_2$ and there exist $\mathcal{A}$ and $(\mathcal{A}_\alpha : \alpha < \omega_1)$ as in Theorem 4.1. We shall show that $\exp(\Psi(\mathcal{A}))$ is not pseudocompact.

Let $F = \langle F_n : n \in \omega \rangle \subseteq \exp(\Psi(\mathcal{A}))$ be given by $F_n = \{(n,m) : m \leq n\}$. We claim that F has no accumulation point in $\exp(\Psi(\mathcal{A}))$. Suppose L is such an accumulation point. Then, since F is a sequence of pairwise disjoint finite subsets of $\triangle$, $L \subseteq \mathcal{A}$.

If $|L| < \omega_2$, there exists a total function f below the diagonal which is almost disjoint from every element of L. Then $L \in (\Psi(\mathcal{A}) \setminus \operatorname{cl} f)^+$ but $F_n \notin (\Psi(\mathcal{A}) \setminus \operatorname{cl} f)^+$ for every $n \in \omega$, a contradiction.

Now suppose $|L| = \omega_2$. There exists $\alpha < \omega_1$ such that there exists two distinct $s, t \in \mathcal{A}$ such that $\operatorname{dom} s, \operatorname{dom} t \in \mathcal{A}_\alpha$. Since s, t are distinct, it follows that $\operatorname{dom}(s) \neq \operatorname{dom}(t)$, and since $\mathcal{A}_\alpha$ is an almost disjoint family, $\operatorname{dom} s \cap \operatorname{dom} t \subseteq k$ for some $k \in \omega$. Then

$$L \in (\{s\} \cup \{s \setminus \{(n,m) : m \leq n < k\}\})^- \cap (\{t\} \cup \{t \setminus \{(n,m) : m \leq n < k\}\})^-,$$

but no element of the sequence F is a member of the latter open set. $\square$

Let $\mathcal{A}$ be an AD family. For the convenience of the reader we repeat the definition of the *Mathias forcing* $\mathbb{M}(\mathcal{A})$ associated with $\mathcal{A}$. The base set is the collection of all $p = (s_p, F_p)$ such that

(1) there is $n_p \in \omega$ such that $s_p : n_p \longrightarrow 2$, and
(2) $F_p \in [\mathcal{A}]^{<\omega}$,

ordered by $p = (s_p, F_p) \leq q = (s_q, F_q)$ if

(1) $s_q \subseteq s_p$ (hence $n_q \leq n_p$), $F_q \subseteq F_p$, and
(2) if $B \in F_q$, then $B \cap s_p^{-1}(1) \subseteq n_q$.

Given $p = (s_p, F_p) \in \mathbb{M}(\mathcal{A})$, we call s_p the *stem of* p and F_p the *side condition of* p. The *length of* p is $\operatorname{len}(p) = n_p$. If $G \subseteq \mathbb{M}(\mathcal{A})$ is a generic filter, the *generic real of* $\mathbb{M}(\mathcal{A})$ is defined as $A_{gen} = \bigcup \{i \mid \exists (s,F) \in G\, (s(i) = 1)\}$. The following lemma is well-known and easy to prove:

Lemma 4.3. *Let $\mathcal{A}$ be an AD family, $G \subseteq \mathbb{M}(\mathcal{A})$ a generic filter and A_{gen} the generic real.*

(1) A_{gen} is an infinite subset of ω.
(2) A_{gen} is almost disjoint from every element of $\mathcal{A}$.
(3) For every $X \in [\omega]^\omega \cap V$, if $X \in \mathcal{I}(\mathcal{A})^+$, then $A_{gen} \cap X$ is infinite.

By Fun we denote the set of all functions $f : \omega \longrightarrow \omega$ such that $f \subseteq \triangle$. Define PFun as the set of all functions g such that there is $A \in [\omega]^\omega$ for which $g : A \longrightarrow \omega$ and $g \subseteq \triangle$. Note that if $f, g \in \mathrm{PFun}$ then f and g are almost disjoint if and only if the set $\{n \in \operatorname{dom}(f) \cap \operatorname{dom}(g) \mid f(n) = g(n)\}$ is finite.

Definition 4.4. Define $\mathfrak{ie}$ as the smallest size of a family $\mathcal{F} \subseteq \mathrm{PFun}$ such that for every $g \in \mathrm{Fun}$ there is $f \in \mathcal{F}$ such that $|f \cap g| = \omega$.

The cardinal invariant $\mathfrak{ie}$ is closely related (though not equal) to the invariant $\operatorname{cov}^*(\mathcal{ED}_{\mathrm{fin}})$ defined in [16]. If $X \in [\omega]^\omega$ and $n \in \omega$, we let $X(n)$ be the n-th element of X.

Definition 4.5. Let $X \in [\omega]^{\omega}$ and $\mathcal{B} \subseteq \mathrm{PFun}$. Define the forcing $\mathbb{E}_{\triangle}(\mathcal{B}, X)$ as the set of all $p = (s_p, n_p, F_p)$ with the following properties:

(1) $n_p \in \omega$, $F_p \in [\mathcal{B}]^{<\omega}$.
(2) $s_p : X \cap n_p \longrightarrow \omega$ and $s_p \subseteq \triangle$.
(3) $2|F_p| \leq n_p$.

Let $p = (s_p, n_p, F_p)$, $q = (s_q, n_q, F_q) \in \mathbb{E}_{\triangle}(\mathcal{B})$, we define $p \leq q$ if the following conditions hold:

(1) $n_q \leq n_p$, $F_q \subseteq F_p$ and $s_q \subseteq s_p$.
(2) If $f \in F_q$ and $i \in \mathrm{dom}\, f \cap (X \cap (n_p \setminus n_q))$, then $s_p(i) \neq f(i)$.

Given $p = (s_p, n_p, F_p) \in \mathbb{E}_{\triangle}(\mathcal{B}, X)$, we call s_p the *stem of* p and F_p the *side condition of* p. Define the *length of* p as $\mathrm{len}(p) = n_p$. By $\mathbb{E}_{\triangle}$ we will denote $\mathbb{E}_{\triangle}(\mathrm{Fun}, \omega)$. If $G \subseteq \mathbb{E}_{\triangle}(\mathcal{B}, X)$ is a generic filter, the *generic real of* $\mathbb{E}_{\triangle}(\mathcal{B}, X)$ is defined as $f_{gen} = \bigcup\{s \mid \exists (s, n, F) \in G\}$. The analogue of Lemma 4.3 is the following:

Lemma 4.6. *Let* $X \in [\omega]^{\omega}$, $\mathcal{B} \subseteq \mathrm{PFun}$ *and* f_{gen} *the generic real of* $\mathbb{E}_{\triangle}(\mathcal{B}, X)$.

(1) $f_{gen} : X \longrightarrow \omega$ *and* $f_{gen} \subseteq \triangle$.
(2) f_{gen} *is almost disjoint from every element of* $\mathcal{B}$.
(3) If $g \in \mathrm{PFun} \cap V$ *is such that* $\mathrm{dom}(g) \subseteq X$ *and* $g \in \mathcal{I}(\mathcal{B})^{+}$ *(where* $\mathcal{I}(\mathcal{B})$ *is the ideal generated by* $\mathcal{B}$*), then* $f_{gen} \cap g$ *is infinite.*

Let $\mathbb{P}$ be a partial order. Recall that a set $L \subseteq \mathbb{P}$ is *linked* if every $p, q \in L$ are compatible. $\mathbb{P}$ is σ-linked if $\mathbb{P}$ is the union of countably many linked sets. The following establishes that $\mathbb{E}_{\triangle}(\mathcal{B}, X)$ is σ-linked:

Lemma 4.7. *Let* $X \in [\omega]^{\omega}$ *and* $\mathcal{B} \subseteq \mathrm{Fun}$. *Let* $p = (s_p, n_p, F_p)$, $q = (s_q, n_q, F_q) \in \mathbb{E}_{\triangle}(\mathcal{B}, X)$. *If* $s_p = s_q$ *and* $4|F_p|, 4|F_q| \leq n_p$ *then* $r = (s_p, n_p, F_p \cup F_q)$ *extends both* p *and* q.

Proof. Let $p = (s_p, n_p, F_p)$, $q = (s_q, n_q, F_q) \in \mathbb{E}_{\triangle}(\mathcal{B}, X)$ with $s = s_p = s_q$. We first find a finite partial function $t \subseteq \triangle$ with the following properties:

(1) $s \subseteq t$.
(2) For every $f \in F_p \cup F_q$ and $i \in \mathrm{dom}(t) \setminus \mathrm{dom}(s)$, we have that $t(i) \neq f(i)$.
(3) $|t| \geq 2|F_p \cup F_q|$.

We can find such t since $4|F_p|, 4|F_q| \leq n_p$. It follows that $r = (t, \mathrm{dom}(t), F_p \cup F_q)$ is an extension of both p and q. □

Lemma 4.8. $\mathbb{E}_{\triangle}(\mathcal{B}, X)$ *is* σ*-linked.*

Proof. For every $n \in \omega$ and $s : X|_n \longrightarrow \omega$ with $s \subseteq \triangle$, define

$$L(s, n) = \{q \mid \exists p \leq q \; p = (s_p, n_p, F_p) \;\; n_p = n, \;\; s_p = s \text{ and } 4|F_p| \leq n_p\}.$$

Clearly each $L(s, n)$ is linked by the previous lemma and

$$\mathbb{E}_{\triangle}(\mathcal{B}, X) = \bigcup\{L(s, n) : n \in \omega, s \subseteq \triangle, s \in \omega^{X|_n}\}. \quad \square$$

The following result was inspired by Lemma 5.1 of A. Miller's [21]:

Proposition 4.9. *Let $n \in \omega$, $s : n \longrightarrow \omega$ with $s \subseteq \triangle$. Let $D \subseteq \mathbb{E}_\triangle$ be an open dense set. There is an antichain $Z \in [D]^{<\omega}$ such that for every $p = (s, n, F_p) \in \mathbb{E}_\triangle$, there is $q \in Z$ such that p and q are compatible.*

Proof. Let $A = \{r_m \mid m \in \omega\} \subseteq D$ be a maximal antichain (note that A is countable since $\mathbb{E}_\triangle$ is σ-linked and therefore c.c.c.), let $k = \frac{n}{2}$ in case n is even and $k = \frac{n-1}{2}$ in case n is odd.

Assume the proposition is false, so for every $m \in \omega$, there is $p_m = (s, n, F_m) \in \mathbb{E}_\triangle$ such that $p_m \perp r_i$ for each $i \leq m$. As $|F_m| \leq k$ we can assume that each F_m has size k, let $F_m = \{f_i^m\}_{i<k}$. We may view $B = \{F_m \mid m \in \omega\}$ as a subset of Fun^k. Since Fun^k is a compact space, we can find an accumulation point $F = \{g_i\}_{i<k}$ of B. Let $p = (s, n, F)$, since A is a maximal antichain, there is $j \in \omega$ such that p and r_j are compatible. Let $q = (t, l, G)$ be a common extension of both of them. Since F is an accumulation point of B, there is $m > l, j$ such that $f_i^m \restriction l = g_i \restriction l$ for every $i < k$. Let $\overline{p}_m = (t, l, F_m)$ and note that $\overline{p}_m \leq p_m$. It follows that $\overline{p}_m$ and q are compatible, in particular, p_m and q are compatible, which implies that p_m and r_j are compatible, which is a contradiction. □

For the rest of the section, we fix sets $\{D_\gamma \mid \gamma \in \omega_1\}$, H, E and a function R with the following properties:

(1) $\{H, E\} \cup \{D_\gamma \mid \gamma \in \omega_1\}$ is a partition of ω_2.
(2) For every $\gamma \in \omega_1$, we have that $|D_\gamma| = |H| = |E| = \omega_2$.
(3) $R : \bigcup\limits_{\gamma\in\omega_1} D_\gamma \longrightarrow H$ is a bijective function such that $\alpha < R(\alpha)$ for every $\alpha \in \bigcup\limits_{\gamma\in\omega_1} D_\gamma$.

Then we define a finite support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha \leq \omega_2 \rangle$ as follows:

(1) If $\alpha \in E$, then $\mathbb{P}_\alpha \Vdash$ "$\dot{\mathbb{Q}}_\alpha = \mathbb{E}_\triangle$".
(2) For every $\gamma \in \omega_1$ and $\xi \in D_\gamma$, let $\dot{A}_\gamma^\xi$ be a name for the $(\mathbb{M}(\mathcal{A}_\gamma^\xi), V_\xi)$-generic real (where $\mathcal{A}_\gamma^\xi = \{\dot{A}_\gamma^\eta \mid \eta \in \xi \cap D_\gamma\}$ and V_ξ is the extension by $\mathbb{P}_\xi$).
(3) If $\alpha \in D_\gamma$ (with $\gamma \in \omega_1$), then $\mathbb{P}_\alpha \Vdash$ "$\dot{\mathbb{Q}}_\alpha = \mathbb{M}(\mathcal{A}_\gamma^\alpha)$".
(4) Given $\xi \in H$, let $\gamma \in \omega_1$ and $\beta \in D_\gamma$ such that $\xi = R(\beta)$, let $\dot{f}_\xi$ be a name for the $(\mathbb{E}_\triangle(\mathcal{B}_\xi, A_\gamma^\beta), V_\xi)$-generic real (where $\mathcal{B}_\xi = \{\dot{f}_\eta \mid \eta \in \xi \cap H\}$).
(5) If $\alpha \in H$, with $(R(\beta) = \alpha$ and $\beta \in D_\gamma)$ then $\mathbb{P}_\alpha \Vdash$ "$\dot{\mathbb{Q}}_\alpha = \mathbb{E}_\triangle(\mathcal{B}_\alpha, A_\gamma^\beta)$".

If $p \in \mathbb{P}_\alpha$ and $\dot{x}$ is a $\mathbb{P}_\alpha$-name for a condition of $\dot{\mathbb{Q}}_\alpha$, we denote by $p^\frown \dot{x}$ the condition $r \in \mathbb{P}_{\alpha+1}$ such that $r \restriction \alpha = p$ and $r(\alpha) = \dot{x}$.

We will need to develop some combinatorial tools for our forcing in order to prove the main result. Given $\alpha \leq \omega_2$, we say that a condition $p \in \mathbb{P}_\alpha$ is *pure* if there is $n \in \omega$ such that for every $\xi \in \mathrm{dom}(p)$, the following holds:

(1) If $\xi \in D_\gamma$ (for some $\gamma \in \omega_1$), then there is $s_\xi \in 2^n$ and $J_\xi \in [D_\gamma \cap \xi]^{<\omega}$, $J_\xi \subseteq \mathrm{dom}(p)$ such that $p(\xi) = (s_\xi, \{\dot{A}_\gamma^\eta \mid \eta \in J_\xi\})$.
(2) If $\xi \in H$ and β is such that $R(\beta) = \xi$, then $\beta \in \mathrm{dom}(p)$.
(3) If $\xi \in H$, (let β such that $R(\beta) = \xi$), then there is $z_\xi : s_\beta^{-1}(1) \longrightarrow \omega$ with $z_\xi \subseteq \triangle$ and $J_\xi \in [H \cap \xi]^{<\omega}$, $J_\xi \subseteq \mathrm{dom}(p)$ such that $p(\xi) = (z_\xi, n, \{\dot{f}_\eta \mid \eta \in J_\xi\})$ and $4|J_\xi| \leq n$ (where s_β is defined as in point 1).
(4) If $\xi \in E$, then there is $m_\xi \in \omega$, $z_\xi : m_\xi \longrightarrow \omega$ with $z_\xi \subseteq \triangle$ and $\dot{J}$ such that $p(\xi) = (z_\xi, m_\xi, \dot{J})$ and there is k_ξ such that $4k_\xi \leq m_\xi$ and $\mathbb{P}_\xi$-names $\rho_0, \ldots, \rho_k$ for functions such that $\dot{J} = \{(\rho_0, \mathbb{1}_{\mathbb{P}_\xi}), \ldots, (\rho_{k_\xi-1}, \mathbb{1}_{\mathbb{P}_\xi})\}$

Given a pure condition p, $\mathrm{len}(p)$ denotes the size of the first coordinate of p.

In the above definition, recall that $\dot{A}_\gamma^\xi$ is the name for the $(\mathbb{M}(\mathcal{A}_\gamma^\xi), V_\xi)$-generic real and $\dot{f}_\xi$ is the name for the $(\mathbb{E}_\triangle(\mathcal{B}_\xi), V_\xi)$-generic real. An important difference between points 3 and 4 is that in point 4 we may have $m_\xi \neq n$. We call n *the height of* p. One of the purposes of pure conditions is to avoid (as much as possible) the use of names and use real objects.

Lemma 4.10. *Pure conditions are dense in* $\mathbb{P}_\alpha$.

Proof. We prove the lemma by induction on α. The cases where $\alpha = 0$ or α is limit are straightforward, so we focus on the successor case. Assume the lemma is true for α, we will prove it is also true for $\alpha + 1$. Let $p \in \mathbb{P}_{\alpha+1}$, we may assume that $\alpha \in \mathrm{dom}(p)$.

Case. $\alpha \in E$.

First, we find $p_1 \leq p \restriction \alpha$ such that there are $m_\alpha \in \omega$, $z_\alpha : m_\alpha \longrightarrow \omega$ with $z_\alpha \subseteq \triangle$ and $\dot{L}$ such that $p_1 \Vdash$ "$p(\alpha) = (z_\alpha, m_\alpha, \dot{L})$". By extending p and p_1, we may even assume that $p_1 \Vdash$ "$4|\dot{L}| \leq m_\alpha$". So we may find $p_2 \leq p_1$, $k_\xi \leq \frac{m_\xi}{4}$ and names $\rho_0, \ldots, \rho_{k_\xi - 1}$ such that $p_2 \Vdash \dot{L} = \{\rho_0, \ldots, \rho_{k_\xi-1}\}$. Let $\dot{J} = \{(\rho_0, \mathbb{1}_{\mathbb{P}_\xi}), \ldots, (\rho_{k_\xi-1}, \mathbb{1}_{\mathbb{P}_\xi})\}$. By the inductive hypothesis, let $q \leq p_2$ be a pure condition. Define $\overline{q} \in \mathbb{P}_{\alpha+1}$ such that the following holds:

(1) $\overline{q} \restriction \alpha = q$.
(2) $\overline{q}(\alpha) = (z_\alpha, m_\alpha, \dot{J})$.

It is easy to see that $\overline{q}$ is a pure extension of p.

Case. $\alpha \in D_\gamma$ (for some $\gamma \in \omega_1$).

First, we find $p_1 \leq p \restriction \alpha$ such that there are $m \in \omega$, $s \in 2^m$ and $J_\alpha \in [D_\alpha \cap \alpha]^{<\omega}$ such that $p_1 \Vdash$ "$p(\alpha) = (s, \{\dot{A}_\gamma^\eta \mid \eta \in J_\alpha\})$", we may assume that $J_\alpha \subseteq \mathrm{dom}(p_1)$. By the inductive hypothesis, let $q \leq p_1$ be a pure condition, let n witnessing that q is pure, without lost of generality, we may assume that $m < n$. Let $s_\alpha \in 2^n$ such that $s_\alpha \restriction m = s$ and $s_\alpha(i) = 0$ for every $i \in [m, n)$. Define $\overline{q} \in \mathbb{P}_{\alpha+1}$ such that the following holds:

(1) $\overline{q} \restriction \alpha = q$.
(2) $\overline{q}(\alpha) = (s_\alpha, \{\dot{A}_\gamma^\eta \mid \eta \in J_\alpha\})$.

It is easy to see that $\overline{q}$ is a pure extension of p.

Case. $\alpha \in H$.

First, we find $p_1 \leq p \restriction \alpha$ such that there are $m \in \omega$, $s : m \longrightarrow \omega$ with $s \subseteq \triangle$ and $J_\alpha \in [H \cap \alpha]^{<\omega}$ such that $p_1 \Vdash$ "$p(\alpha) = (s, m, \{\dot{f}_\eta \mid \eta \in J_\alpha\})$", we may also assume that $4|J_\alpha| < m$ and that $J_\alpha \subseteq \mathrm{dom}(p_1)$. By the inductive hypothesis, let $q \leq p_1$ be a pure condition, let n witnessing that q is pure, without lost of generality, we may assume that $m < n$ and $J_\alpha \subseteq \mathrm{dom}(q)$. Let $z_\alpha : n \longrightarrow \omega$ such that $z_\alpha \subseteq \triangle$, $z_\alpha \restriction m = s$ and $z_\alpha(i) \neq z_\xi(i)$ for every $i \in [m, n)$ and $\xi \in J_\alpha$. Define $\overline{q} \in \mathbb{P}_{\alpha+1}$ such that the following holds:

(1) $\overline{q} \restriction \alpha = q$.
(2) $\overline{q}(\alpha) = (z_\alpha, n, \{\dot{f}_\eta \mid \eta \in J_\alpha\})$.

It is easy to see that $\overline{q}$ is a pure extension of p. $\square$

Lemma 4.11. *Let $\alpha \leq \omega_2$, $p \in \mathbb{P}_\alpha$ a pure condition and $m \in \omega$. There is $q \in \mathbb{P}_\alpha$ with the following properties:*

(1) $q \leq p$.
(2) q *is pure.*
(3) If $\beta \in \mathrm{dom}\,(q)$ *then* $m \leq \mathrm{len}\,(q\,(\beta))$.

Proof. We prove the lemma by induction on α. The cases where $\alpha = 0$ or α is limit are straightforward, so we focus on the successor case. Assume the lemma is true for α, we will prove it is also true for $\alpha + 1$. Let $p \in \mathbb{P}_{\alpha+1}$, we may assume that $\alpha \in \mathrm{dom}\,(p)$.

Case. $\alpha \in E$.

Suppose $p\,(\alpha) = (z_\alpha, m_\alpha, \dot{J})$. In case that $m \leq m_\alpha$, we apply the inductive hypothesis to $p \restriction \alpha$ and we are done. Assume that $m_\alpha < m$. By the inductive hypothesis, we may find $q \leq p \restriction \alpha$ such that the following holds:

(1) q is pure.
(2) If $\beta \in \mathrm{dom}\,(q)$ then $\mathrm{len}\, q(\beta) \geq m$.
(3) For every $j < k_\xi$ there is $w_j : m \longrightarrow \omega$ such that $q \Vdash$ "$\rho_j \restriction m = w_j$".

We now define $s : m \longrightarrow \omega$, with $s \subseteq \triangle$ such that $z_\alpha \subseteq s$ and $s\,(i) \neq w_j\,(i)$ for every $i \in (m_\alpha, m]$ and $j < n$. It is clear that $q^\frown(s, m, \dot{J})$ has the desired properties.

Case. $\alpha \in D_\gamma$ (for some $\gamma \in \omega_1$).

Suppose $p\,(\alpha) = (s_\alpha, \{\dot{A}^\eta_\gamma : \eta \in \dot{J}_\alpha\})$ and n is such that $s_\alpha : n \longrightarrow 2$. By the inductive hypothesis, we may find $q \leq p \restriction \alpha$ such that the following holds:

(1) q is pure.
(2) If $\beta \in \mathrm{dom}\,(p)$ then $\mathrm{len}\, q(\beta) \geq \max\{m, n\}$.

Let k be the height of q. We now define $z : k \longrightarrow 2$ such that $s_\alpha \subseteq z$ and $z\,(i) = 0$ for every $i \in [n, k)$. It is clear that $q^\frown(z, \dot{J}_\alpha)$ has the desired properties.

Case. $\alpha \in H$.

Similar to the previous cases. □

Definition 4.12. Let $\alpha \leq \omega_2$ and $p \in \mathbb{P}_\alpha$ a pure condition. We say that p has the *descending condition* if for every $\beta_1, \beta_2 \in \mathrm{dom}\,(p) \cap E$, if $\beta_1 < \beta_2$, then $\mathrm{len}\,(p\,(\beta_1)) \geq \mathrm{len}\,(p\,(\beta_2))$.

Using the previous lemma and induction, we get the following:

Lemma 4.13. *For every $\alpha \leq \omega_2$, the pure conditions with the descending condition are dense.*

Proof. We prove the lemma by induction on α. The cases where $\alpha = 0$ or α is limit are straightforward, so we focus on the successor case. Assume the lemma is true for α, we will prove it is also true for $\alpha + 1$. Let $p \in \mathbb{P}_{\alpha+1}$ be a pure condition, we may assume that $\alpha \in \mathrm{dom}\,(p)$. In case $\alpha \notin E$, there is nothing to do, so assume that $\alpha \in E$.

Let $p(\alpha) = (s, n, \dot{J})$, by the inductive hypothesis and Lemma 4.11, we can find $q \in \mathbb{P}_\alpha$ such that $q \leq p \restriction \alpha$, q is pure with the descending condition and all the stems in q have size larger than n. It is clear that $q^\frown (s, n, \dot{J})$ is the condition we are looking for. □

Although pure conditions are nice to work with, we will need to deal with non-pure conditions for some arguments. We will develop the tools needed in order to do this. First, we recall a standard forcing lemma that will often be used implicitly (for a proof, see Lemma 1.19 in the first chapter of [24]):

Lemma 4.14. *Let $\mathbb{P}$ be a partial order, $A = \{p_\alpha \mid \alpha \in \kappa\} \subseteq \mathbb{P}$ a maximal antichain and $\{\dot{x}_\alpha \mid \alpha \in \kappa\}$ be a set of $\mathbb{P}$-names. There is a $\mathbb{P}$-name $\dot{y}$ such that $p_\alpha \Vdash$ "$\dot{y} = \dot{x}_\alpha$" for every $\alpha \in \kappa$.*

Given $A \in [E]^{<\omega}$, a function $K : A \longrightarrow \omega^{<\omega}$ is said to be *suitable* if $K(\alpha) \subseteq \triangle$ for every $\alpha \in A$. We say that a condition $q \in \mathbb{P}_{\omega_2}$ *follows* a suitable K if the following holds:

(1) $A \subseteq \text{dom}(q)$.
(2) If $\alpha \in A$, then $q \restriction \alpha \Vdash$ "$q(\alpha) = (K(\alpha), |K(\alpha)|, \dot{F})$" (for some $\dot{F}$).

Definition 4.15. Let $A \in [E]^{<\omega}$. We say that $p \in \mathbb{P}_\alpha$ has the *A-descending condition* if the following holds:

(1) For every $\beta_1, \beta_2 \in (\text{dom}(p) \setminus A) \cap E$, if $\beta_1 < \beta_2$, then $p \restriction \beta_2 \Vdash$ "$\text{len}(p(\beta_1)) \geq \text{len}(p(\beta_2))$".
(2) For every $\beta_1, \beta_2 \in \text{dom}(p) \cap H$, if $\beta_1 < \beta_2$, then $p \restriction \beta_2 \Vdash$ "$\text{len}(p(\beta_1)) \geq \text{len}(p(\beta_2))$".
(3) For every $\gamma \in \omega_1$ and for every $\beta_1, \beta_2 \in \text{dom}(p) \cap D_\gamma$, if $\beta_1 < \beta_2$, then $p \restriction \beta_2 \Vdash$ "$\text{len}(p(\beta_1)) \geq \text{len}(p(\beta_2))$".
(4) If $\beta = \min(\text{dom}(p))$, then there is $s \in \omega^{<\omega}$ such that s is the stem of $p(\beta)$ (i.e., the stem of $p(\beta)$ is a real object, not just a name) and for every $\eta \in \text{dom}(p) \setminus A$, we have that $p \restriction \eta \Vdash$ "$\text{len}(p(\beta)) \geq \text{len}(p(\eta))$".

Notice that this new notion does not clash with our previous terminology, since pure conditions with the descending condition (essentially) satisfy the $\emptyset$-descending condition. We now introduce the following notions:

Definition 4.16. Let $\alpha \in \omega_2$, $A \in [E \cap \alpha]^{<\omega}$ and $K : A \longrightarrow \omega^{<\omega}$ be suitable. We define $\mathbb{P}_\alpha^K$ as the set of all $p \in \mathbb{P}_\alpha$ such that the following conditions hold:

(1) p follows K.
(2) p satisfies the A-descending condition.
(3) For every $\beta \in \text{dom}(p) \cap (H \cup E)$, if $p(\beta) = (\dot{s}, \dot{m}, \dot{F})$, then $p \restriction \beta \Vdash$ "$4|\dot{F}| \leq \dot{m}$".

The following result is similar to Lemma 4.11:

Lemma 4.17. *Let $\alpha \leq \omega_2$, $A \in [E \cap \alpha]^{<\omega}$, $K : A \longrightarrow \omega^{<\omega}$ be suitable, $p \in \mathbb{P}_\alpha^K$ and $m \in \omega$. There is q such that the following holds:*

(1) $q \in \mathbb{P}_\alpha^K$.
(2) $\text{dom}(q) = \text{dom}(p)$.
(3) $q \leq p$.
(4) If $\beta \in A$, *then* $q(\beta) = p(\beta)$.
(5) If $\beta \in \text{dom}(q) \setminus A$ *then* $q \restriction \beta \Vdash \text{len}(q(\beta)) = \max\{m, \text{len}(p(\beta))\}$.

Proof. Note that the last point already implies that q satisfies the A-descending condition. We proceed by induction, the cases $\alpha = 0$ and α is limit are immediate. Assume the lemma is true for α, we will now prove it for $\alpha + 1$. We may assume that $\alpha \in \text{dom}(p)$.

Case. $\alpha \notin H \cup E$.

Note that in particular, $\alpha \notin A$. Let $p \restriction \alpha \Vdash p(\alpha) = \left(\dot{s}, \dot{F}\right)$, by the inductive hypothesis, there is $q \leq p \restriction \alpha$ as in the lemma. Let $\dot{k}$ be a $\mathbb{P}_\alpha$-name for a natural number, such that $q \Vdash$ "$\dot{s} : \dot{k} \longrightarrow 2$". Let $\dot{z}$ be a $\mathbb{P}_\alpha$-name such that q forces the following:

(1) $\text{dom}(\dot{z}) = \max\{m, \dot{k}\}$.
(2) $\dot{s} \subseteq \dot{z}$.
(3) If $i \in \text{dom}(\dot{z}) \setminus \text{dom}(\dot{s})$, then $\dot{z}(i) = 0$.

It is clear that $q^\frown(\dot{z}, \dot{F})$ is the condition we were looking for.

Case. $\alpha \in H$.

Let $\alpha \in H$, $\gamma \in \omega_1$ and $\beta \in D_\gamma$ such that $R(\beta) = \alpha$. Let $p \in \mathbb{P}^K_{\alpha+1}$ with $\alpha \in \text{dom}(p)$. By the inductive hypothesis, we may assume that $p \restriction \alpha$ satisfy the properties in the conclusion of the lemma. Let $p \restriction \alpha \Vdash p(\alpha) = (\dot{s}, \dot{k}, \dot{F})$ and find $\dot{n}$ a $\mathbb{P}_\alpha$-name for $\max\{\dot{k}, m\}$. Let $\dot{z}$ be a $\mathbb{P}_\alpha$-name for a partial function forced to have the following properties:

(1) $\dot{z} \subseteq \triangle$.
(2) $\dot{s} \subseteq \dot{z}$.
(3) $\text{dom}(\dot{z}) = \dot{A}^\beta_\gamma \cap \dot{n}$
(4) for all $i \in \text{dom}(\dot{z})$, if $i \notin \text{dom}(\dot{s})$, then $\dot{z}(i) = \min\left\{j \mid \forall g \in \dot{F}\,(g(i) \neq j)\right\}$.

It is clear that $p \restriction \alpha^\frown \left(\dot{z}, \dot{n}, \dot{F}\right)$ has the desired properties.

Case. $\alpha \in E$ and $\alpha \notin A$.

Similar to the previous case.

Case. $\alpha \in E$ and $\alpha \in A$.

Let $A_1 = A \setminus \{\alpha\}$ and $K_1 = K \restriction A_1$. By the inductive hypothesis (applied to $p \restriction \alpha$ and K_1) let $q \leq p \restriction \alpha$ as in the lemma. It is easy to see that $q^\frown p(\alpha)$ has the desired properties. $\square$

We will need the following result, which is the generalization of Proposition 4.9 for the iteration:

Lemma 4.18. *Let $\alpha \leq \omega_2$, $D \subseteq \mathbb{P}_\alpha$ an open dense set, $A \in [E \cap \alpha]^{<\omega}$ and $K : A \longrightarrow \omega^{<\omega}$ suitable. If $p \in \mathbb{P}^K_\alpha$, then there is q with the following properties:*

(1) $q \in \mathbb{P}^K_\alpha$
(2) $q \leq p$.
(3) If $\beta \in A$, then $q(\beta) = p(\beta)$.
(4) There is an antichain $L \in [D]^{<\omega}$ such that for every $r \leq q$, if r follows K, then r is compatible with an element of L.

Proof. We prove the lemma by induction on α. The case where $\alpha = 0$ is clear. We will now prove it for $\alpha + 1$.

Case. $\alpha \notin A$.

Define $\overline{D}$ as the set of all $q \in \mathbb{P}_\alpha$ for which there exists $\overline{q} \in \mathbb{P}_{\alpha+1}$ with the following properties:

(1) $\overline{q} \restriction \alpha = q$.
(2) $\overline{q} \in D$.
(3) $q \Vdash$ "$\overline{q}(\alpha) \leq p(\alpha)$".
(4) There is $m_q \in \omega$ such that $q \Vdash$ "$\operatorname{len}(\overline{q}(\alpha)) = m_q$".
(5) In case $\alpha \in H \cup E$, if $q \Vdash \overline{q}(\alpha) = (\dot{s}, m_q, \dot{F})$, then $q \Vdash$ "$4|\dot{F}| \leq m_q$".

It is easy to see that $\overline{D}$ is an open dense subset of $\mathbb{P}_\alpha$. By the inductive hypothesis, there is $\overline{p} \leq p \restriction \alpha$ as in the lemma, let $L \in [\overline{D}]^{<\omega}$ an antichain such that for every $q \leq \overline{p}$, if q follows K, then q is compatible with an element of L. Let $L = \{q_i \mid i < k\}$ for some $k \in \omega$. For every $i < k$, fix $\overline{q}_i \in D$ as in the definition of $\overline{D}$. Let $\beta_0 = \min(\operatorname{dom}(p))$, we now find $m \in \omega$ such that $m > \operatorname{len}(p(\beta_0))$ as well as $m > m_{q_i}$ for every $i < k$. Since L is an antichain, we can find $\dot{x}$ a $\mathbb{P}_\alpha$-name for an element of $\dot{\mathbb{Q}}_\alpha$ with the following properties:

(1) $q_i \Vdash$ "$\dot{x} = \overline{q}_i(\alpha)$" for every $i < k$.
(2) $r \Vdash$ "$\dot{x} = p(\alpha)$" for every r incompatible with every q_i.

We now apply Lemma 4.17 to find p_1 with the following properties:

(1) $p_1 \in \mathbb{P}_\alpha^K$.
(2) $p_1 \leq \overline{p}$.
(3) $\operatorname{dom}(p_1) = \operatorname{dom}(\overline{p})$.
(4) If $\gamma \in A$, then $p_1(\gamma) = \bar{p}(\gamma)$.
(5) If $\beta \in \operatorname{dom}(p_1) \setminus A$, then $p_1 \restriction \beta \Vdash$ "$\operatorname{len}(p_1(\beta)) = \max\{m, len(\overline{p}(\beta))\}$".

Let $q = p_1{}^\frown \dot{x}$. We claim that q has the desired properties. In order to prove that $q \in \mathbb{P}^K_{\alpha+1}$, we only need to prove that q has the A-descending condition (the other properties are true by definition). Note that p_1 forces that the length of the stem of $\dot{x}$ is at most m (since $p \in \mathbb{P}^K_{\alpha+1}$, then $\operatorname{len}(p(\alpha))$ is forced to be at most $\operatorname{len}(p(\beta_0))$, which is smaller than m). Since the length of the stem in all the elements of $\operatorname{dom}(p_1) \setminus A$ is at least m, it follows that q has the A-descending condition. Clearly $q \leq p$ and if $\beta \in A$, then $q(\beta) = p(\beta)$.

Finally, let $L_1 = \{\overline{q}_i \mid i < k\} \subseteq D$ and let $r \leq q$ be a condition following K. We need to prove that r is compatible with an element of L_1. Since $r \restriction \alpha \leq q \restriction \alpha$ and it follows K, we know there is $q_i \in L$ such that $r \restriction \alpha$ and q_i are compatible. We claim that r and $\overline{q}_i$ are compatible.

Let $r_1 \in \mathbb{P}_\alpha$ be a common extension of both $r \restriction \alpha$ and q_i. Define $\overline{r} = r_1{}^\frown r(\alpha)$, we will prove that $\overline{r}$ extends both r and $\overline{q}_i$. Clearly $\overline{r} \leq r$ and in order to show that $\overline{r} \leq \overline{q}_i$, we only need to prove that $r_1 \Vdash$ "$r(\alpha) \leq \overline{q}_i(\alpha)$". Since $r_1 \leq q_i$, we have that $r_1 \Vdash$ "$\dot{x} = \overline{q}_i(\alpha)$". We also know that $r \restriction \alpha \Vdash$ "$r(\alpha) \leq \dot{x}$", we conclude that $r_1 \Vdash$ "$r(\alpha) \leq \overline{q}_i(\alpha)$" and we are done.

Case. $\alpha \in A$ (in particular, $\alpha \in E$).

Let $s = K(\alpha)$ and $n = |s|$. In this way, we have that $p \restriction \alpha \Vdash$ "$p(\alpha) = (s, n, \dot{F})$" for some $\dot{F}$. Let $G \subseteq \mathbb{P}_\alpha$ be a generic filter with $p \restriction \alpha \in G$. In $V[G]$, we define the set $\overline{D} = \{\dot{x}[G] \mid \exists q \leq p \restriction \alpha\, (q \in G \wedge q^\frown \dot{x} \in D)\}$. It is easy to see that $\overline{D}$ is an open dense subset of $\mathbb{E}_\triangle$. By the Proposition 4.9, there is $Z \in [\overline{D}]^{<\omega}$ an antichain such that for every $x = (s, n, J) \in \mathbb{E}_\triangle$, there is $z \in Z$ such that x and z are compatible.

Back in V, define B as the set of all $r \in \mathbb{P}_\alpha$ with the following properties:

(1) Either r and $p \restriction \alpha$ are incompatible, or
(2) There are $k \in \omega$ and $Y^r = \{\dot{x}_i^r \mid i < k\}$ such that $r \Vdash$ "$\dot{Z} = \{\dot{x}_i^r[\dot{G}] \mid i < k\}$" and $r^\frown \dot{x}_i^r \in D$ for every $i < k$.

It is easy to see that B is an open dense subset of $\mathbb{P}_\alpha$. Let $K_1 = K \restriction \alpha$. We apply the inductive hypothesis with $p \restriction \alpha$, B and K_1. In this way, there are q and L with the following properties:

(1) $q \leq p \restriction \alpha$.
(2) $q \in \mathbb{P}_\alpha^{K_1}$.
(3) If $\beta \in A \setminus \{\alpha\}$, then $q(\beta) = p(\beta)$.
(4) $L \in [B]^{<\omega}$ is an antichain.
(5) For every $q' \leq q$, if q' follows K_1, then q_1 is compatible with an element of L.

We now define $L_1 = \{r^\frown \dot{x}_i^r \mid r \in L \wedge \dot{x}_i^r \in Y^r\}$, note that L_1 is a finite antichain of D. Define $\overline{q} = q^\frown p(\alpha)$, we claim that $\overline{q}$ and L_1 have the desired properties. Clearly $\overline{q} \in \mathbb{P}_{\alpha+1}^K$. Now, let $q_1 \leq \overline{q}$ that follows K. Since $q_1 \restriction \alpha \leq \overline{q} \restriction \alpha = q$ and $q_1 \restriction \alpha$ follows K_1, we know that there is $r \in L$ compatible with $q_1 \restriction \alpha$. Let $q_2 \leq q_1 \restriction \alpha, r$ and note that $q_2 \Vdash$ "$\dot{Z} = \{\dot{x}_i^r[\dot{G}] \mid i < k\}$", hence (without lost of generality), there is i such that q_2 forces that $q_1(\alpha)$ and $\dot{x}_i^r$ are compatible (recall that $q_1(\alpha)$ is forced to be of the form $(s, n, \dot{J})$ since q_1 follows K). It follows that q_1 and $r^\frown \dot{x}_i^r$ are compatible.

Finally, we consider the case when α is a limit ordinal and the proposition is true for every $\beta < \alpha$. This case is similar to the one where $\alpha \notin A$. We first find $\beta < \alpha$ such that $A, \mathrm{dom}(p) \subseteq \beta$. Define $\overline{D}$ as the set of all $q \in \mathbb{P}_\beta$ such that there is $\overline{q} \in \mathbb{P}_\alpha$ with the following properties:

(1) $\overline{q} \restriction \beta = q$.
(2) $\overline{q} \in D$.
(3) If $\xi \in (\mathrm{dom}(\overline{q}) \setminus \beta) \cap (H \cup E)$ and $\overline{q} \restriction \xi \Vdash \overline{q}(\xi) = (\dot{z}, \dot{m}, \dot{J})$ then $\overline{q} \restriction \xi \Vdash$ "$4\left|\dot{J}\right| \leq \dot{m}$".
(4) $\overline{q} \restriction [\beta, \alpha)$ has the decreasing condition.
(5) There is $n_{\overline{q}}$ such that for every $\xi \in \mathrm{dom}(\overline{q}) \setminus \beta$, the condition $\overline{q} \restriction \xi \Vdash$ "$\mathrm{len}(\overline{q}(\xi)) \leq n_{\overline{q}}$".

It is easy to see that $\overline{D}$ is an open dense subset of $\mathbb{P}_\beta$ (it is dense by Lemma 4.13). By the induction hypothesis, there are $q \leq p$ following K and an antichain $L = \{q_i \mid i < k\} \subseteq \overline{D}$ such that for every $r \leq q$ that follows K, r is compatible with an element of L. For every $i < k$, choose $\overline{q}_i \in D$ witnessing that $q_i \in \overline{D}$. Find $n \in \omega$ such that $n > n_{\overline{q}_i}$ for every $q_i \in L$. By Lemma 4.17, we may assume that all of the stems in $\mathrm{dom}(q) \setminus A$ are forced to be larger than n. Let $B_i = \mathrm{dom}(\overline{q}_i)$ for every $i < k$. We now define a condition $\widehat{q} \in \mathbb{P}_\alpha$ with the following properties:

(1) $\widehat{q} \restriction \beta = q$.
(2) $\mathrm{dom}(\widehat{q}) = \mathrm{dom}(q) \cup \bigcup_{i<k} B_i$
(3) For every $i < k$ and $\xi \in B_i$, we have that $q_i \restriction \xi \Vdash$ "$\widehat{q}(\xi) = \overline{q}_i(\xi)$".
(4) For every $i < k$ and $\xi \in \alpha$ such that $\xi \notin \beta \cup B_i$, we have that $q_i \restriction \xi \Vdash$ "$\widehat{q}(\xi) = 1_{\dot{\mathbb{Q}}_\xi}$" (where $1_{\dot{\mathbb{Q}}_\xi}$ is the name of the largest condition).
(5) If $r \in \mathbb{P}_\beta$ is incompatible with every $q_i \in L$ and $\xi \in \bigcup_{i<k} B_i$, then $r \restriction \xi \Vdash$ "$\widehat{q}(\xi) = 1_{\dot{\mathbb{Q}}_\xi}$"

Let $L_1 = \{\overline{q}_i \mid i < k\}$, we will show that $\widehat{q}$ and L_1 have the desired properties. It is easy to see that $\widehat{q} \in \mathbb{P}^K_\alpha$. Now, let $r \leq \widehat{q}$ that follows K. Clearly, $r \restriction \beta$ extends q and follows K, so there is $i < k$ such that q_i is compatible with r. It is easy to see that r is compatible with $\overline{q}_i$. □

We can now prove the following:

Proposition 4.19. *There is a model of ZFC such that:*

(1) $\mathfrak{c} = \omega_3$*.*
(2) $\mathfrak{ie} = \omega_2$*.*
(3) There are families $\{\mathcal{A}_\gamma \mid \gamma \in \omega_1\}$*,* $\mathcal{B} = \{f_\alpha \mid \alpha \in \omega_2\}$ *such that:*

- *(a)* $\mathcal{A}_\gamma \subseteq [\omega]^\omega$ *is a MAD family of size* ω_2 *(for every* $\gamma \in \omega_1$*).*
- *(b)* $\mathcal{B} \subseteq \mathrm{PFun}$ *is a MAD family.*
- *(c) If* $\pi : \mathrm{PFun} \longrightarrow [\omega]^\omega$ *is the function defined by* $\pi(f) = \mathrm{dom}(f)$*, then* $\pi \restriction \mathcal{B} : \mathcal{B} \longrightarrow \bigcup_{\gamma \in \omega_1} \mathcal{A}_\gamma$ *is bijective.*

Proof. We start with a ground model such that $V \models \mathfrak{c} = \omega_3$ and we will force with $\mathbb{P}_{\omega_2}$. Let $G \subseteq \mathbb{P}_{\omega_2}$ be a generic filter. It is easy to see that $V[G] \models \mathfrak{c} = \omega_3$. For every $\gamma \in \omega_1$, let $\mathcal{A}_\gamma = \left\{A^\alpha_\gamma \mid \alpha \in D_\gamma\right\}$. We have the following:

Claim. *Let* $\gamma \in \omega_1$*.*

(1) $\mathcal{A}_\gamma \subseteq [\omega]^\omega$ *is a MAD family of size* ω_2*.*
(2) For every $X \in V[G]$*, if* $X \in \mathcal{I}(\mathcal{A}_\gamma)^+$*, then the set* $\{\alpha \in D_\gamma \mid \left|X \cap A^\alpha_\gamma\right| = \omega\}$ *has size* ω_2*.*

The claim follows easily by Lemma 4.3. A more interesting fact is the following:

Claim. $V[G] \models \bigcap_{\gamma \in \omega_1} \mathcal{I}(\mathcal{A}_\gamma) = [\omega]^{<\omega}$.

Let $\dot{X}$ be a $\mathbb{P}_{\omega_2}$-name for an infinite subset of ω. Let $M \in V$ be a countable elementary submodel of $\mathsf{H}((2^{\omega_3})^+)$ such that $\dot{X}, \mathbb{P}_{\omega_2} \in M$. Choose $\gamma \in \omega_1 \setminus M$, we will show that $\dot{X}$ is forced to be in $\mathcal{I}(\mathcal{A}_\gamma)^+$. In fact, we will prove that $\dot{X}$ will have infinite intersection with every element of $\mathcal{A}_\gamma$. Note that $D_\gamma \cap M = \emptyset$ since $\gamma \notin M$ (recall that $\{D_\eta \mid \eta \in \omega_1\} \in M$ since $\mathbb{P}_{\omega_2} \in M$).

Let $\xi \in D_\gamma$, $k \in \omega$ and $p \in \mathbb{P}_{\omega_2}$ (in general, $p \notin M$). We must find an extension of p forcing that $\dot{X}$ and A^ξ_γ intersect beyond k. We may assume that $\xi \in \mathrm{dom}(p)$, p is pure and has the descending condition. Let n be the height of p. We may also assume that $n > k$. For technical reasons, assume that $0 \in \mathrm{dom}(p)$. Let $B = \mathrm{dom}(p) \cap M$ and $A = B \cap E$. Note that $p \in \mathbb{P}^K_\alpha$, where K is the suitable function on A defined by "$K(\alpha)$ is the first coordinate of the triple $p(\alpha)$". Let $\mathrm{dom}(p) = \{\alpha_0, ..., \alpha_m\}$ where $\alpha_i < \alpha_j$ whenever $i < j$.

Claim. *There is* $\overline{p} \in M \cap \mathbb{P}_{\omega_2}$ *such that for every* $i \leq m$*, the following holds:*

(1) $\overline{p}$ *is pure of height* n*.*
(2) $\mathrm{dom}(\overline{p}) = \{\delta_0, ..., \delta_m\}$ *(where* $\delta_i < \delta_j$ *whenever* $i < j$*) and* $B \subseteq \mathrm{dom}(\overline{p})$*.*
(3) $\overline{p} \in \mathbb{P}^K_{\omega_2}$*.*
(4) If $\alpha_i \in B$*, then* $\delta_i = \alpha_i$*.*
(5) If $\alpha_i \notin B$*, then* $\delta_i < \alpha_i$*.*
(6) $\alpha_i \in E$ *if and only if* $\delta_i \in E$*.*

(7) $\alpha_i \in H$ *if and only if* $\delta_i \in H$.
(8) *For every* $\eta \in M \cap \omega_1$, *if* $\alpha_i \in D_\eta$ *then* $\delta_i \in D_\eta$.
(9) *For every* $j \leq m$, *if* $\alpha_i, \alpha_j \in \bigcup_{\eta \in \omega_1} D_\eta$ *then* α_i, α_j *are in the same element of the partition if and only if* δ_i, δ_j *are in the same element of the partition.*
(10) *If* $\alpha_i \in H$, *then the following holds:*

(a) *If* $p(\alpha_i) = \left(s_{\alpha_i}, n, \{\dot{f}_\mu : \mu \in J^p_{\alpha_i}\}\right)$, *then* $\overline{p}(\delta_i) = \left(s_{\alpha_i}, n, \{\dot{f}_\mu : \mu \in J^{\overline{p}}_{\delta_i}\}\right)$ *(i.e. the stem of* $p(\alpha_i)$ *and* $\overline{p}(\delta_i)$ *is the same).*
(b) *For every* $j < i$, *we have that* $\alpha_j \in J^p_{\alpha_i}$ *if and only if* $\delta_j \in J^{\overline{p}}_{\delta_i}$.

(11) *If* $\alpha_i \in D_\eta$ *for some* $\eta < \omega_1$, *then the following holds:*

(a) *If* $p \restriction \alpha_i \Vdash$ *"*$p(\alpha_i) = \left(s_{\alpha_i}, \{\dot{A}^\mu_\eta : \mu \in J^p_{\alpha_i}\}\right)$*", then* $\overline{p} \restriction \delta_i \Vdash$ *"*$\overline{p}(\delta_i) = \left(s_{\alpha_i}, \{\dot{A}^\mu_\eta : \mu \in J^{\overline{p}}_{\delta_i}\}\right)$*" (i.e. the stem of* $p(\alpha_i)$ *and* $\overline{p}(\delta_i)$ *is the same).*
(b) *For every* $j < i$, *we have that* $\alpha_j \in J^p_{\alpha_i}$ *if and only if* $\delta_j \in J^{\overline{p}}_{\delta_i}$ *(where* $\alpha_i \in D_\eta$ *and* $\delta_i \in D_{\eta'}$*).*

(12) *If* $\alpha_i \in E$, *then the following holds:*

(a) *If* $p(\alpha_i) = \left(s_{\alpha_i}, m_{\alpha_i}, J^p_{\alpha_i}\right)$, *then* $\overline{p}(\delta_i) = \left(s_{\alpha_i} m_{\alpha_i}, J^{\overline{p}}_{\delta_i}\right)$ *(i.e. the stems of* $p(\alpha_i)$ *and* $\overline{p}(\delta_i)$ *are the same).*
(b) *If* $\alpha_i \in M$, *then* $J^p_{\alpha_i} \cap M = J^{\overline{p}}_{\delta_i} \cap M$ *(recall that in this case,* $\alpha_i = \delta_i$*).*

The claim is almost an immediate consequence of the elementarity of M, point 5 is the only one that requires us being slightly more careful. For every $\alpha_i \notin B$, we define the following:

(1) $\xi^0_i = \max(B) \cap \alpha_i$ (this is well defined since $0 \in B$).
(2) $\xi^1_i = \min(M \cap (\omega_2 + 1) \setminus \alpha_i)$.

Note that $\xi^0_i, \xi^1_i \in M$ and $\xi^0_i < \alpha_i < \xi^1_i$. The claim then follows by applying elementarity and requiring that $\xi^0_i < \delta_i < \xi^1_i$. Since $\delta_i \in M$ and is smaller that ξ^1_i, it follows that $\delta_i < \alpha_i$.

Let $\overline{p}$ be as in the claim. We now define

$$D = \left\{r \in \mathbb{P}_{\omega_2} \mid \exists l_r \in \omega \left(r \Vdash \text{"} l_r = \min\left(\dot{X} \setminus n\right) \text{"}\right)\right\}.$$

Clearly $D \subseteq \mathbb{P}_{\omega_2}$ is an open dense subset and $D \in M$. Since $\overline{p} \in \mathbb{P}^K_{\omega_2}$, applying Lemma 4.18, there is $q \leq \overline{p}$ as in the lemma. We may even assume that $q \in M$. Note that in general, q might not be pure (we could extend it to a pure condition, but it might not follow K anymore). Let $L \in [D]^{<\omega}$ such that for every $r \leq q$, if r follows K, then r is compatible with an element of L. Let $Z = \{l_r \mid r \in L\}$ and note that $Z \cap n = \emptyset$. It is clear that if $r \in L$, then $r \Vdash$ "$Z \cap \dot{X} \neq \emptyset$". Let $n_1 = \max(Z) + 1$.

We now define the condition p_Z with the following properties:

(1) $\mathrm{dom}(p_Z) = \mathrm{dom}(p)$.
(2) For every $\eta \in \mathrm{dom}(p_Z)$, the following holds:

(a) If $\eta \notin D_\gamma$, then $p_Z(\eta) = p(\eta)$.
(b) Let $\eta \in D_\gamma$ with $\eta \neq \xi$. If $p(\eta) = \left(s^p_\eta, \{\dot{A}^\mu_\gamma : \mu \in J^p_\eta\}\right)$ define $s^{pz}_\eta : n_1 \longrightarrow 2$ such that $s^p_\eta \subseteq s^{pz}_\eta$ and $s^{pz}_\eta(i) = 0$ for every $i \in [n, n_1)$. Let $p_Z(\eta) = \left(s^{pz}_\eta, \{\dot{A}^\mu_\gamma : \mu \in J^p_\eta\}\right)$.

(c) If $p(\xi) = \left(s_\xi^p, \{\dot{A}_\gamma^\mu : \mu \in J_\xi^p\}\right)$ define $s_\xi^{pz} : n_1 \longrightarrow 2$ such that $s_\xi^p \subseteq s_\xi^{pz}$ and $s_\xi^{pz}(i) = 1$ for every $i \in [n, n_1)$. Let $p_Z(\xi) = \left(s_\xi^{pz}, \{\dot{A}_\gamma^\mu : \mu \in J_\xi^p\}\right)$.

Note that $p_Z \Vdash$ "$Z \subseteq A_\gamma^\xi$". Since $J_\xi^p \subseteq \text{dom}(p \restriction \xi)$, it is follows from (b) that $p_Z \leq p$. We now define the condition r as follows:

(1) $\text{dom}(r) = \text{dom}(p_Z) \cup \text{dom}(q)$.
(2) If $\eta \in \text{dom}(q) \setminus \text{dom}(p_Z)$, then $r(\eta) = q(\eta)$.
(3) Let $\eta \in \text{dom}(p_Z)$, so $\eta = \alpha_i$ for some $i \leq m$. We have the following:

(a) Assume $\alpha_i \in D_{\gamma'}$ with $\gamma' \notin M$ (so $\eta \notin \text{dom}(q)$), define $r(\alpha_i) = p_Z(\alpha_i)$ (note that this will be the case when $\gamma' = \gamma$).
(b) Assume $\alpha_i \in D_{\gamma'}$ with $\gamma' \in M$. Let $p_Z(\alpha_i) = \left(s_{\alpha_i}^{pz}, \{\dot{A}_{\gamma'}^\mu : \mu \in J_{\alpha_i}^{pz}\}\right)$ and $q \restriction \delta_i \Vdash q(\delta_i) = \left(\dot{t}_{\delta_i}^q, \{\dot{A}_{\gamma'}^\mu : \mu \in \dot{J}_{\delta_i}^q\}\right)$ (since q is not pure, $\dot{t}_{\delta_i}^q$ and $\dot{J}_{\delta_i}^q$ might be names and not actual objects). Define $r(\alpha_i) = \left(\dot{t}_{\delta_i}^q, \{\dot{A}_{\gamma'}^\mu : \mu \in J_{\alpha_i}^{pz} \cup \dot{J}_{\delta_i}^q\}\right)$. In here, note that $\dot{t}_{\delta_i}^q$ is a $\mathbb{P}_{\delta_i}$-name, since $\delta_i \leq \alpha_i$ it is also a $\mathbb{P}_{\alpha_i}$-name, so the definition at least makes sense.
(c) Assume $\alpha_i \in H$. Let $p_Z(\alpha_i) = \left(s_{\alpha_i}^{pz}, n, \{\dot{f}_\mu : \mu \in J_{\alpha_i}^{pz}\}\right)$, $q(\delta_i) = \left(\dot{t}_{\delta_i}^q, \dot{m}_{\delta_i}^q, \{\dot{f}_\mu : \mu \in \dot{J}_{\delta_i}^q\}\right)$, and $r(\alpha_i) = \left(\dot{t}_{\delta_i}^q, \dot{m}_{\delta_i}^q, \{\dot{f}_\mu : \mu \in \dot{J}_{\delta_i}^q \cup J_{\alpha_i}^{pz}\}\right)$.
(d) Assume $\alpha_i \in E$ and $\alpha_i \notin \text{dom}(q)$. Define $r(\alpha_i) = p_Z(\alpha_i)$.
(e) Assume $\alpha_i \in E$ and $\alpha_i \in \text{dom}(q)$ (so $\delta_i = \alpha_i$ and $\alpha_i \in A$). Let $p_Z(\alpha_i) = \left(s_{\alpha_i}^{pz}, n, \dot{J}_{\alpha_i}^{pz}\right)$ and note that in here we have that $q(\delta_i) = \left(s_{\alpha_i}^{pz}, n, \dot{J}_{\delta_i}^q\right)$ (this is because $\alpha_i \in A$, so $q(\delta_i) = \overline{p}(\delta_i)$). Define $r(\delta_i) = \left(s_{\alpha_i}^{pz}, n, \dot{J}_{\delta_i}^q \cup \dot{J}_{\alpha_i}^{pz}\right)$.

A key remark is that in r, we do mot change the stem of the coordinates that are in E. It might not be immediately obvious that r is a condition, since the "size requirement" may fail in the coordinates of E or H. We will show that this is not the case.

Claim. *Let $\eta \in \text{dom}(r)$.*

(1) $r \restriction \eta \in \mathbb{P}_\eta$.
(2) $r \restriction \eta \Vdash$ "$r(\eta) \in \dot{\mathbb{Q}}_\eta$".
(3) $r \restriction \eta \leq q \restriction \eta$.
(4) $r \restriction \eta \Vdash$ "$r(\eta) \leq q(\eta)$ ".

We will prove the claim. Note that points 3 and 4 are trivial once we know that $r \restriction \eta$ is a condition. We proceed by induction, it is enough to show that if $r \restriction \eta \in \mathbb{P}_\eta$ and $r \restriction \eta \leq q \restriction \eta$, then $r \restriction \eta \Vdash$ "$r(\eta) \in \dot{\mathbb{Q}}_\eta$". Furthermore, this is clear whenever $\eta \in \text{dom}(q) \setminus \text{dom}(p_Z)$, $\eta \notin H \cup E$ or $\eta \in E \setminus \text{dom}(q)$. We focus on the other cases. From now on, $\eta \in \text{dom}(p_Z)$, so we may assume that $\eta = \alpha_i$ for some $i \leq m$.

Case. $\alpha_i \in H$.

In here, $p_Z(\alpha_i) = \left(s_{\alpha_i}^p, n, \{\dot{f}_\mu : \mu \in J_{\alpha_i}^p\}\right)$, $q(\delta_i) = \left(\dot{t}_{\delta_i}^q, \dot{m}_{\delta_i}^q, \{\dot{f}_\mu : \mu \in \dot{J}_{\delta_i}^q\}\right)$, and $r(\alpha_i) = (\dot{t}_{\delta_i}^q, \dot{m}_{\delta_i}^q, \{\dot{f}_\mu : \mu \in \dot{J}_{\delta_i}^q \cup J_{\alpha_i}^p\})$. As $\overline{p}(\delta_i) = \left(s_{\alpha_i}^p, n, \{\dot{f}_\mu : \mu \in J_{\delta_i}^{\overline{p}}\}\right)$ and since $q \leq \overline{p}$, we get that $q \restriction \delta_i \Vdash$ "$n \leq \dot{m}_{\delta_i}^q$". Furthermore, $q \restriction \delta_i \Vdash$ "$4\left|\dot{J}_{\delta_i}^q\right| \leq \dot{m}_{\delta_i}^q$". We also know that $4\left|J_{\alpha_i}^p\right| \leq n$, (since p is pure), hence $q \restriction \delta_i \Vdash_{\mathbb{P}_{\delta_i}}$ "$4\left|\dot{J}_{\delta_i}^q\right|, 4\left|J_{\alpha_i}^{pz}\right| \leq \dot{m}_{\delta_i}^q$". Since $r \restriction \alpha_i \leq r \restriction \delta_i \leq q \restriction \delta_i$, $\mathbb{P}_{\delta_i}$ is completely embedded into $\mathbb{P}_{\alpha_i}$ and the formula is absolute for transitive models of ZFC, we get that $r \restriction \alpha_i \Vdash_{\mathbb{P}_{\alpha_i}}$ "$4\left|\dot{J}_{\delta_i}^q\right|, 4\left|J_{\alpha_i}^{pz}\right| \leq \dot{m}_{\delta_i}^q$", so $r(\alpha)$ is forced to be a condition by Lemma 4.7.

Case. $\alpha_i \in \operatorname{dom} q \cap E$.

In here, $p_Z(\alpha_i) = \left(s^p_{\alpha_i}, m_{\alpha_i}, \dot{J}^p_{\alpha_i}\right)$, $q(\alpha_i) = \overline{p}(\alpha_i) = \left(s^p_{\alpha_i}, m_{\alpha_i}, \dot{J}^q_{\alpha_i}\right)$ and $r(\alpha_i) = \left(s^p_{\alpha_i}, m_{\alpha_i}, \dot{J}^q_{\alpha_i} \cup \dot{J}^p_{\alpha_i}\right)$. Clearly, $r \restriction \alpha_i \Vdash 4\left|\dot{J}^q_{\alpha_i}\right|, 4\left|\dot{J}^p_{\alpha_i}\right| \leq m_\alpha$ since any condition forces this statement, so $r(\alpha)$ is forced to be a condition by Lemma 4.7.

We now know that r is indeed a condition and that $r \leq q$. Note that r follows K.

We will now prove that $r \leq p_Z$. Let $\alpha_i \in \operatorname{dom}(p_Z)$, assume that we know that $r \restriction \alpha_i \leq p_Z \restriction \alpha_i$, we will prove that $r \restriction \alpha_i \Vdash$ "$r(\alpha_i) < p_Z(\alpha_i)$ ". We proceed by cases:

Case. $\alpha_i \in D_{\gamma'}$ with $\gamma' \notin M$.

This case is immediate by the definition.

Case. $\alpha_i \in D_{\gamma'}$ with $\gamma' \in M$ and $\alpha_i \in \operatorname{dom}(q)$ (hence $\delta_i = \alpha_i$).

In here, we have that

$$p_Z(\alpha_i) = \left(s^{p_Z}_{\alpha_i}, \{\dot{A}^\mu_{\gamma'} : \mu \in J^{p_Z}_{\alpha_i}\}\right),\ q(\alpha_i) = \left(\dot{t}^q_{\alpha_i}, \{\dot{A}^\mu_{\gamma'} : \mu \in \dot{J}^q_{\alpha_i}\}\right)$$

and $r(\alpha_i) = \left(\dot{t}^q_{\alpha_i}, \{\dot{A}^\mu_{\gamma'} : \mu \in J^{p_Z}_{\alpha_i} \cup \dot{J}^q_{\alpha_i}\}\right)$. Since $r \leq q$, we have that $r \leq \overline{p}$, so $r \restriction \alpha_i \Vdash$ "$s^{p_Z}_{\alpha_i} \subseteq \dot{t}^q_{\alpha_i}$" (in this case, $s^{\overline{p}}_{\alpha_i} = s^p_{\alpha_i} = s^{p_Z}_{\alpha_i}$).

Now, let $\alpha_j \in J^{p_Z}_{\alpha_i}$ (recall that the stem of $r(\alpha_j)$ is $\dot{t}^q_{\alpha_j}$). We need to prove that $r \restriction \alpha_i \Vdash$ " $\left(\dot{t}^q_{\alpha_i}\right)^{-1}(1) \cap A^{\alpha_j}_{\gamma'} \subseteq n$". Let $\dot{m}_{\alpha_i}, \dot{m}_{\alpha_j}$ such that $q \restriction \alpha_i \Vdash$ "$\dot{t}^q_{\alpha_i} : \dot{m}_{\alpha_i} \longrightarrow 2$" and $q \restriction \alpha_j \Vdash$ "$\dot{t}^q_{\alpha_j} : \dot{m}_{\alpha_j} \longrightarrow 2$". Since q satisfies the A-descending condition, we know that $q \restriction \alpha_i \Vdash$ "$\dot{m}_{\alpha_j} \geq \dot{m}_{\alpha_i}$". Since $q \restriction \alpha_j \Vdash$ "$A^{\alpha_j}_{\gamma'} \cap \dot{m}_{\alpha_j} = \left(\dot{t}^q_{\alpha_j}\right)^{-1}(1)$ ", we get that $q \restriction \alpha_i \Vdash$ "$A^{\alpha_j}_{\gamma'} \cap \dot{m}_{\alpha_i} = \left(\dot{t}^q_{\alpha_j}\right)^{-1}(1) \cap \dot{m}_{\alpha_i}$". Since $r \leq \overline{p}$, we know that $r \Vdash$ "$A^{\alpha_i}_{\gamma'} \cap A^{\alpha_j}_{\gamma'} \subseteq n$". In particular, $\dot{t}^q_{\alpha_i}$ is forced to be disjoint with $\dot{A}^{\alpha_j}_{\gamma'} \setminus n$, so we get that $r \Vdash$ " $\left(\dot{t}^q_{\alpha_i}\right)^{-1}(1) \cap \left(\dot{t}^q_{\alpha_j}\right)^{-1}(1) \subseteq n$", hence $r \restriction \alpha_i \Vdash$ " $\left(\dot{t}^q_{\alpha_i}\right)^{-1}(1) \cap A^{\alpha_j}_{\gamma'} \subseteq n$", which is what we wanted to prove.

Case. $\alpha_i \in D_{\gamma'}$ with $\gamma' \in M$ and $\alpha_i \notin \operatorname{dom}(q)$ (so $\delta_i < \alpha_i$).

Here we have $p_Z(\alpha_i) = \left(s^{p_Z}_{\alpha_i}, \{\dot{A}^\mu_{\gamma} : \mu \in J^{p_Z}_{\alpha_i}\}\right)$, $q(\delta_i) = \left(\dot{t}^q_{\delta_i}, \{A^\mu_{\gamma'} : \mu \in \dot{J}^q_{\delta_i}\}\right)$ and $r(\alpha_i) = (\dot{t}^q_{\delta_i}, \{A^\mu_{\gamma'} : \mu \in \dot{J}^q_{\delta_i} \cup J^{p_z}_{\alpha_i}\})$. Since $r \leq q$, we have that $r \leq \overline{p}$, so $r \restriction \delta_i \Vdash$ "$s^{p_Z}_{\alpha_i} \subseteq \dot{t}^q_{\delta_i}$" (recall that $s^{p_Z}_{\alpha_i}$ is the stem of $\overline{p}(\delta_i)$).

Now, let $\alpha_j \in J^{p_Z}_{\alpha_i}$ (recall that the stem of $r(\alpha_j)$ is $\dot{t}^q_{\delta_j}$). We need to prove that $r \restriction \alpha_i \Vdash$ " $\left(\dot{t}^q_{\delta_i}\right)^{-1}(1) \cap A^{\alpha_j}_{\gamma'} \subseteq n$". Let $\dot{m}_{\delta_i}, \dot{m}_{\delta_j}$ such that $q \restriction \delta_i \Vdash$ "$\dot{t}^q_{\delta_i} : \dot{m}_{\delta_i} \longrightarrow 2$" and $q \restriction \delta_j \Vdash$ "$\dot{t}^q_{\delta_j} : \dot{m}_{\delta_j} \longrightarrow 2$". Since q satisfies the $\operatorname{dom} K$-descending condition, we know that $q \restriction \delta_i \Vdash$ "$\dot{m}_{\delta_j} \geq \dot{m}_{\delta_i}$". Since $q \restriction \delta_j \Vdash$ "$A^{\delta_j}_{\gamma'} \cap \dot{m}_{\delta_j} = \left(\dot{t}^q_{\delta_j}\right)^{-1}(1)$ ", we get that $q \restriction \delta_i \Vdash$ "$A^{\delta_j}_{\gamma'} \cap \dot{m}_{\delta_i} = \left(\dot{t}^q_{\delta_j}\right)^{-1}(1) \cap \dot{m}_{\delta_i}$". Since $r \leq \overline{p}$, we know that $r \Vdash$ "$A^{\delta_i}_{\gamma'} \cap A^{\delta_j}_{\gamma'} \subseteq n$". In particular, $\dot{t}^q_{\delta_i}$ is forced to be disjoint with $\dot{A}^{\delta_j}_{\gamma'} \setminus n$, so we get that $r \Vdash$ " $\left(\dot{t}^q_{\alpha_i}\right)^{-1}(1) \cap \left(\dot{t}^q_{\delta_j}\right)^{-1}(1) \subseteq n$", hence $r \restriction \alpha_i \Vdash$ " $\left(\dot{t}^q_{\delta_i}\right)^{-1}(1) \cap A^{\alpha_j}_{\gamma'} \subseteq n$", which is what we wanted to prove.

Case. $\alpha_i \in H$ and $\alpha_i \in \operatorname{dom}(q)$ (so $\alpha_i = \delta_i$).

Here $p_Z(\alpha_i) = \left(s^{p_Z}_{\alpha_i}, n, \{\dot{f}_\mu : \mu \in J^{p_Z}_{\alpha_i}\}\right)$, $q(\delta_i) = \left(\dot{t}^q_{\delta_i}, \dot{m}^q_{\delta_i}, \{\dot{f}_\mu : \mu \in \dot{J}^q_{\delta_i}\}\right)$ and $r(\alpha_i) = (\dot{t}^q_{\delta_i}, \dot{m}^q_{\delta_i}, \{\dot{f}_\mu : \mu \in \dot{J}^q_{\delta_i} \cup J^{p_Z}_{\alpha_i}\})$. Since $r \leq q$, we have that $r \leq \overline{p}$, so $r \restriction \alpha_i \Vdash$ "$s^{p_Z}_{\alpha_i} \subseteq \dot{t}^q_{\delta_i}$".

Now, let $\alpha_j \in J^{pz}_{\alpha_i}$ (recall that the stem of $r(\alpha_j)$ is $\dot{t}^q_{\delta_j}$). We need to prove that $r \restriction \alpha_i \Vdash$ "$\dot{t}^q_{\delta_i} \cap \dot{f}_{\alpha_j} \subseteq n \times n$". Since q satisfies the dom K-descending condition, we know that $q \restriction \delta_i \Vdash$ "$\dot{m}_{\delta_j} \geq \dot{m}_{\delta_i}$". Since $q \restriction \alpha_j \Vdash$ "$\dot{f}_{\alpha_j} \restriction \dot{m}_{\delta_j} = \dot{t}^q_{\delta_j}$", we get that $q \restriction \delta_i \Vdash$ "$\dot{f}_{\alpha_j} \restriction \dot{m}_{\delta_i} = \dot{t}^q_{\delta_j}$". Since $r \leq \overline{p}$, we know that $r \Vdash$ "$\dot{f}_{\alpha_i} \cap \dot{f}_{\delta_j} \subseteq n \times n$". In particular, $\dot{t}^q_{\delta_i}$ is forced to be disjoint with $\dot{f}_{\alpha_j}$ above n, so we get that $r \Vdash$ "$\dot{t}^q_{\delta_i} \cap \dot{t}^q_{\delta_j} \subseteq n \times n$", hence $r \restriction \alpha_i \Vdash$ "$\dot{t}^q_{\delta_i} \cap \dot{f}_{\alpha_j} \subseteq n \times n$", which is what we wanted to prove.

Case. $\alpha_i \in H$ and $\alpha_i \notin \operatorname{dom}(q)$ (so $\delta_i < \alpha_i$).

Here $p_Z(\alpha_i) = \left(s^{pz}_{\alpha_i}, n, \{\dot{f}_\mu : \mu \in J^{pz}_{\alpha_i}\}\right)$, $q(\delta_i) = \left(\dot{t}^q_{\delta_i}, \dot{m}^q_{\delta_i}, \{\dot{f}_\mu : \mu \in \dot{J}^q_{\delta_i}\}\right)$ and $r(\alpha_i) = (\dot{t}^q_{\delta_i}, \dot{m}^q_{\delta_i}, \{\dot{f}_\mu : \mu \in \dot{J}^q_{\delta_i} \cup J^{pz}_{\alpha_i}\})$. Since $r \leq q$, we have that $r \leq \overline{p}$, so $r \restriction \delta_i \Vdash$ "$s^{pz}_{\alpha_i} \subseteq \dot{t}^q_{\delta_i}$" (recall that $s^{pz}_{\alpha_i}$ is the stem of $\overline{p}(\delta_i)$).

Now, let $\alpha_j \in J^{pz}_{\alpha_i}$ (recall that the stem of $r(\alpha_j)$ is $\dot{t}^q_{\delta_j}$). We need to prove that $r \restriction \alpha_i \Vdash$ "$\dot{t}^q_{\alpha_i} \cap \dot{f}_{\alpha_j} \subseteq n \times n$". Since q satisfies the descending condition, we know that $q \restriction \delta_i \Vdash$ "$\dot{m}_{\delta_j} \geq \dot{m}_{\delta_i}$". Since $q \restriction \alpha_j \Vdash$ "$\dot{f}_{\alpha_j} \restriction \dot{m}_{\delta_j} = \dot{t}^q_{\delta_j}$", we get that $q \restriction \alpha_i \Vdash$ "$\dot{f}_{\alpha_j} \restriction \dot{m}_{\delta_i} = \dot{t}^q_{\delta_j}$". Since $r \leq \overline{p}$, we know that $r \Vdash$ "$\dot{f}_{\alpha_i} \cap \dot{f}_{\delta_j} \subseteq n \times n$". In particular, $\dot{t}^q_{\delta_i}$ is forced to be disjoint with $\dot{f}_{\alpha_j}$ above n, so we get that $r \Vdash$ "$\dot{t}^q_{\delta_i} \cap \dot{t}^q_{\delta_j} \subseteq n \times n$", hence $r \restriction \alpha_i \Vdash$ "$\dot{t}^q_{\delta_i} \cap \dot{f}_{\alpha_j} \subseteq n \times n$", which is what we wanted to prove.

Case. $\alpha_i \in E$ and $\alpha_i \notin \operatorname{dom}(q)$.

This case is immediate from the definition.

Case. $\alpha_i \in E$ and $\alpha_i \in \operatorname{dom}(q)$ (so $\delta_i = \alpha_i$ and $\alpha_i \in A$).

This case is also immediate from the definition.

Having dealt with all the cases, we can finally conclude that $r \leq q, p_Z$. Since r follows K and $r \leq q$, there is $r' \in L$ such that r' and r are compatible. Let $\overline{r}$ be a common extension. Then:

(1) $\overline{r} \Vdash$ "$\dot{X} \cap Z \neq \emptyset$".
(2) $\overline{r} \Vdash$ "$Z \subseteq \dot{A}^\xi_\gamma$" (since $\overline{r} \leq p_Z$).

Hence $\overline{r} \Vdash$ "$\dot{A}^\xi_\gamma \cap \dot{X} \nsubseteq k$", which is what we wanted to prove. We conclude that $V[G] \models \bigcap_{\gamma \in \omega_1} \mathcal{I}(A_\gamma) = [\omega]^{<\omega}$.

Recall, that $\mathcal{B} = \{f_\alpha \mid \alpha \in H\}$.

Claim. *$\mathcal{B}$ is a MAD family of size ω_2.*

It is easy to see that $\mathcal{B}$ is an almost disjoint family of size ω_2, it remains to prove that it is maximal. Let $h \in \text{PFun}$ and $A = \operatorname{dom}(h)$. By the last claim, there is $\gamma \in \omega_1$ such that $A \in \mathcal{I}(\mathcal{A}_\gamma)^+$. In this way, we can find $\beta \in D_\gamma$ such that $C = A \cap A^\beta_\gamma$ is infinite and $h \in V[G_\beta]$, define $h_1 = h \restriction C$ and note that $h_1 \in V[G_{\beta+1}]$. Let $\alpha = R(\beta)$ (so $\beta < \alpha$). First consider the case where $h_1 \in \mathcal{I}(\mathcal{B}_\alpha)$. Then there are $\alpha_1, ..., \alpha_n \in H$ such that $h_1 \subseteq f_{\alpha_1} \cup ... \cup f_{\alpha_n}$, so clearly h_1 has infinite intersection with an f_{α_i}. In case $h_1 \in \mathcal{I}(\mathcal{B}_\alpha)^+$, we will have that $f_\alpha \cap h_1$ is infinite by 4.6.

Finally, we will prove the following:

Claim. $\mathfrak{ie} = \omega_2$.

On the one hand, since $\mathcal{B}$ is MAD, we get that $\mathfrak{ie} \leq \omega_2$. On the other hand, since we are forcing with $\mathbb{E}_\triangle$ cofinally many times, we get that $\omega_2 \leq \mathfrak{ie}$. We conclude that $\mathfrak{ie} = \omega_2$ holds in our model. □

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